COMPUTABLE DIMENSION FOR ORDERED FIELDS

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Abstract. The computable dimension of a structure counts the number of computable copies up to *computable* isomorphism. In this paper, we consider the possible computable dimensions for various classes of computable ordered fields. We show that computable ordered fields with finite transcendence degree are computably stable, and thus have computable dimension 1. We then build computable ordered fields of infinite transcendence degree which have infinite computable dimension, but also such fields which are computably categorical. Finally, we show that 1 is the only possible finite computable dimension for any computable archimedean field.

1. Introduction

Whenever studying the computable content of an algebraic structure, the first step is to present the structures in a computable way. This is often done by coding the elements of the structure's domain by the natural numbers, and ensuring that the functions and relations of the structure are computable. Those structures which admit such a coding are the computable ones. But for all of these, there are multiple ways in which we could have represented the elements of the domain. A fundamental question is whether our choice in this representation is important - might picking different computable copies of the structure give us different computable-theoretic results about the structure? For example, if, as in this paper, the structure is an ordered field, might one computable copy of the field have a computable transcendence basis, while another computable copy have only non-computable transcendence bases?

One way to know whether we need worry about this is to know the computable dimension of the structure.

Definition 1.1. The *computable dimension* of a computable structure $\mathcal{A}$ is the number of distinct computable copies (presentations) of the structure, up to computable isomorphism.

Definition 1.2. A computable structure $\mathcal{A}$ is *computably categorical* if every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. That is, if the computable dimension of $\mathcal{A}$ is 1.

If we require that every isomorphism is computable, then we say the structure is computably stable:

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Definition 1.3. A computable structure $\mathcal{A}$ is *computably stable* if for every computable structure $\mathcal{B}$ classically isomorphic to $\mathcal{A}$, every isomorphism $f : \mathcal{A} \to \mathcal{B}$ is in fact a computable isomorphism.

Computable dimension has been well studied in a variety of structures, including Boolean algebras [5], linear orders [21], trees [12], abelian groups [4], graphs with finite components [3], etc. For many structures, there is an algebraic criterion which determines the computable dimension. For example, if G is a computable ordered abelian group, then G is computably categorical if and only if it has finite rank [8]. For many structures, computable dimension can only be 1 or ω. However, there are examples of partially ordered sets, graphs, lattices and nilpotent groups which have computable dimension n for each $1 \leq n \leq \omega$ (see [5] and [6]).

A notable gap exists for fields, where the question of computable dimension has been particularly difficult to answer. It is known for algebraically closed fields, as well as real closed fields, that the field is computably categorical if and only if the transcendence degree of the field is finite and if the transcendence degree is infinite, then the computable dimension is infinite (see [16] and [19]). Little is known beyond these two simplest of examples, although it is not as simple as looking at transcendence degree: there are fields with infinite transcendence degree which are computably categorical [18], and fields with finite transcendence degree which are not [17]. The question of whether there are any fields with finite computable dimension greater than 1 appears to be settled in the affirmative thanks to upcoming work of Miller, Park, Poonen, Schoutens, and Shlapentokh.

The purpose of this paper is to explore whether insight might be gained by considering *ordered* fields. After reviewing some preliminaries in section 2, we show in section 3 that all computable ordered fields with finite transcendence degree are computably categorical (in fact, computably stable and also relatively computably categorical). In section 4 we consider computable dimension when the transcendence degree of the ordered field is infinite. We show that in the algebraically simplest case the computable dimension is indeed ω, but that there are examples of computably categorical ordered fields as well. While we stop short of determining an algebraic criterion for the computable dimension of ordered fields with infinite transcendence degree, we are able to show that for archimedean fields, the computable dimension cannot be anything besides 1 or ω. This is done in section 5.

We conclude in section 6 with some remaining questions and ideas for further research.

2. Preliminaries

Before considering any computability theory, we review some classical definitions and results from the theory of ordered fields. For a more comprehensive introduction, see the chapter 6 in [10], chapter 11 in [11], or [20]. Throughout the paper, all fields have characteristic 0 (as all ordered fields do) and are countable (as all computable fields are).

Definition 2.1. Let F be a field. An *ordering* on F is a linear order $\leq$ (i.e., a total, transitive, antisymmetric binary relation) such that for all $a, b, c \in F$,

(1) $a \leq b \Longrightarrow a + c \leq b + c$, and
(2) $a \leq b,\ 0 \leq c \Longrightarrow ac \leq bc$.

F is *orderable* if there exists an ordering $\leq$ on F. An *ordered field* is a pair $(F, \leq)$.

Related to ordered fields are real fields.

Definition 2.2. A field F is *formally real* (or simply *real*) provided -1 is not a sum of squares in F.

Classically, a field is orderable if and only if it is formally real. However, this does not hold effectively as there are computable real fields which have no computable ordering. In fact, given any Π^0_1 class C there is a computable real field for which the space of orderings is in Turing degree preserving bijection with C [16].

Given a formally real field F, the algebraic closure of F is no longer real since $x^2 + 1 = 0$ has a root in the algebraic closure, making -1 a square. If we consider a maximal algebraic extension of a real field which is still real, we get a real closure.

Definition 2.3. A field F is *real closed* if F is formally real and no algebraic extension of F is formally real. A *real closure* R_F of a field F is a real closed field which is algebraic over F.

Real closed fields have a unique ordering: the positive elements are simply those which have square roots in the field. This is enough to determine the ordering on the field, as $a < b$ if and only if $b - a$ is positive. Moreover, if we consider a computable real closed field (the field operations are computable), the order can be computably determined: search the field for the square root of either $b - a$ or $a - b$. Thus when dealing with real closed fields, we will freely use order notation (including intervals) even though "$<$" is not part of the signature of real closed fields.

Every formally real field has a real closure, although it need not be unique as a given formally real field may admit multiple orderings. For example, $\mathbb{Q}(\sqrt{2})$ is formally real with two orderings: one in which $0 < \sqrt{2}$ and the other in which $\sqrt{2} < 0$. However, if we consider a formally real field and specify the order (that is, consider an ordered field) then the unique order on the real closed field must extend the order on the base field. Thus uniqueness of the real closure is guaranteed:

Theorem 2.4 (Artin-Schreier)**.** *Any ordered field $(F, \leq)$ has a unique (up to isomorphism) real closure.*

Beyond their unique orderings, real closed fields are nice for a variety of reasons. In a real closed field R, every polynomial of odd degree with coefficients in R has a root in R. Also, $R(\sqrt{-1})$ is necessarily the algebraic closure of R. Real closed fields are also nice from a model theory point of view: the theory of real closed fields (in the language of ordered rings) is a complete, decidable theory (see section 3.3 in [15]). This implies that any two real closed field are elementarily equivalent. Since the real numbers, as an ordered field, are a real closed field, this says that any real closed field shares all the first order algebraic and order-theoretic properties of $\mathbb{R}$ (the "Tarski-Principle").

Another nice property we will make heavy use of is that it is possible to determine the number of roots of a given polynomial in a real closed field. There are multiple ways to do this. One way is to use the fact that the theory of real closed fields is complete and decidable. Alternatively, we can appeal to the purely algebraic Sturm's Theorem, which we now discuss in more detail.

Theorem 2.5 (Sturm's Theorem)**.** *Let $p(x)$ be any polynomial with coefficients in a real closed field R. Then there is a sequence of polynomials*

$$p_0(x), p_1(x), \ldots, p_n(x)$$

such that if $p(\alpha) \neq 0$ and $p(\beta) \neq 0$, then the number of distinct roots of $p(x)$ in the interval $[\alpha, \beta]$ is $V_\alpha - V_\beta$, where V_γ denotes the number of variations in sign of $\{p_0(\gamma), p_1(\gamma), \ldots, p_n(\gamma)\}$.

The polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ can be found effectively. In fact, $p_0(x) = p(x)$, $p_1(x) = p'(x)$ and for $i \geq 2$, $p_i(x)$ is the negative remainder after dividing $p_{i-1}(x)$ by $p_{i-2}(x)$. Since we are concerned with computable real closed fields, we can to calculate $p_i(\gamma)$ for any γ in R and $i = 0, \ldots, n$. Thus we can effectively find V_γ, and as such, the number of roots of $p(x)$ between any α and β which are not roots of $p(x)$. Further, there is a bound (due to Cauchy) on the roots of a given polynomial, so the total number of roots of a given polynomial can be effectively determined. (For a detailed discussion of Sturm's Theorem, and its proof, see [10].)

The real closure of a field is an algebraic extension, but we also consider field extensions which are not algebraic. Recall that for any field F (ordered or otherwise) a set $S \subseteq F$ is *algebraically dependent* if for some $n \in \mathbb{N}$ there is a nonzero polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ and distinct $s_1, \ldots, s_n \in S$ such that $p(s_1, \ldots, s_n) = 0$. S is *algebraically independent* if it is not algebraically dependent. A maximal algebraically independent set in F is called a *transcendence basis* for F over $\mathbb{Q}$. The *transcendence degree* of a field F is the cardinality of some transcendence basis for F. Every non-algebraic extension field of $\mathbb{Q}$ has a transcendence basis over $\mathbb{Q}$, and all transcendence bases of a given field have the same cardinality, so these notions are well defined (see [9]). For any field F, if F is an extension of $\mathbb{Q}$ and has a transcendence basis S, then F is algebraic over the field $\mathbb{Q}(S)$. The field $\mathbb{Q}(S)$ is a *purely transcendental* extension of $\mathbb{Q}$, with a *pure transcendence basis* S. Note that every purely transcendental extension has a pure transcendence basis, but also has transcendence bases which are not pure. (All of this also works for extensions of arbitrary fields instead of $\mathbb{Q}$, but we will only need to consider this simplest of cases.)

Finally, we consider the possibility of infinite elements in an ordered field.

Definition 2.6. For any element a in an ordered field F, define the *absolute value of a* by

$$|a| = \begin{cases} a & \text{if } 0 \leq a \\ -a & \text{if } a < 0 \end{cases}$$

Definition 2.7. An ordered field F is *archimedean* if for all $a \in F$ there is some $n \in \mathbb{N}$ such that $|a| \leq n$.

Now to computability theory. We assume familiarity with the basic ideas from the subject (otherwise, see [22]). Intuitively, an ordered field will be computable if the operations $+$ and $\cdot$ are computable, and the relation $\leq$ is computable. Specifically, we work in the language of ordered rings, so a field F will have a domain $|F|$ and there will be binary function symbols $+_F$ and $\cdot_F$, a binary relation $\leq_F$, and distinguished elements 0_F and 1_F. For F to be a *computable* ordered field, $|F|$ will be a computable subset of $\mathbb{N}$, with $+_F$ and $\cdot_F$ partial computable functions from $|F| \times |F|$ to $|F|$, and $\leq_F \subseteq |F| \times |F|$ a computable relation. Additionally, we are given the elements 0_F and 1_F computably, although these can always be found uniformly by searching through the elements of the field. Thus the subfield $\mathbb{Q}$ (ordered fields have characteristic zero) is computably enumerable (c.e.) for all computable ordered fields. Of course we want F to be an ordered field, so the usual ordered field axioms must be satisfied. Note that since the domain of F is a subset of $\mathbb{N}$, computable ordered fields (and in general all computable algebraic structures) are necessarily countable.

3. Ordered Fields with Finite Transcendence Degree

We show that every computable ordered field with finite transcendence degree is computably stable. We begin by verifying the same result for computable real closed fields.

Lemma 3.1. *Any computable real closed field with finite transcendence degree is computably stable.*

Proof. Let $R = \{a_0, a_1, \ldots\}$ be a computable real closed field with finite transcendence degree. Let $\widehat{R}$ be a computable real closed field isomorphic to R via the (classical) isomorphism f. Without loss of generality, assume $\{a_0, \ldots, a_{k-1}\}$ is a transcendence basis for R and that a_k is the multiplicative identity. Then $\{f(a_0), \ldots, f(a_{k-1})\}$ is a transcendence basis for $\widehat{R}$ and $f(a_k)$ is the multiplicative identity in $\widehat{R}$. Let $E = \mathbb{Q}(a_0, \ldots, a_{k-1}) \subseteq R$. We will show that f is in fact a computable isomorphism.

Note first that we can computably determine $f(t)$ for any $t \in E$. This is possible since we know the finite information $f(a_0), \ldots, f(a_{k-1})$ and $f(a_k)$. Every other element t of E is some arithmetic combination (sum, difference, product, or quotient) of these finitely many elements. Once we find what combination gives us t, we can form that same combination in $\widehat{R}$, using the fact that f is an isomorphism. Now suppose $p(x)$ is a polynomial in $E[x]$, say

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Since $c_0, \ldots, c_n$ are in E, we can effectively find the polynomial

$$\widehat{p}(x) = f(c_0) + f(c_1)x + \cdots + f(c_n)x^n$$

in $f(E)[x]$.

To compute $f(t)$ for $t \in R$, we first search for and find a polynomial $p(x) \in E[x]$ such that $p(t) = 0$. There must be one since R is algebraic over E. Once found, we determine the number of roots of $p(x)$ which lie in R (which is the same as the number of roots of $\widehat{p}(x)$ which lie in $\widehat{R}$). This can be done either by using Sturm's theorem, or the completeness of the theory of real closed fields. Once we know the number of roots of $p(x)$, we simply search through R to find all of them. Using the computable order on R, we find m such that there are exactly m roots of $p(x)$ less than t. Next, we search through $\widehat{R}$ to find all the roots of $\widehat{p}(x)$, and specifically find the root $\widehat{t}$ which is greater than exactly m other roots. Since f is an isomorphism, it must be that $f(t) = \widehat{t}$, which we have now found. □

Every ordered field has a unique (up to isomorphism) real closure. If F is a computable ordered field, then there is a computable presentation R_F of its real closure, and a computable embedding from F to R_F (see [14]). We will use this to prove our result, but we need to know that isomorphisms behave nicely when we pass to real closures. The next lemma is purely algebraic.

Lemma 3.2. *Let F and $\widehat{F}$ be ordered fields and $f : F \to \widehat{F}$ be an isomorphism. Let R and $\widehat{R}$ be real closures of F and $\widehat{F}$ respectively. Then f extends to a unique isomorphism $g : R \to \widehat{R}$.*

Proof. Define g as follows. First, for every $a \in F$, let $g(a) = f(a)$. Now let a be an element of $R \setminus F$. Since R is an algebraic extension of F, there is a polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Say $p(x) = c_0 + c_1 x + \cdots + c_n x^n$, and define $\widehat{p}(x) = f(c_0) + f(c_1)x + \cdots + f(c_n)x^n$. Let $a_0 < a_1 < \cdots < a_m$ be the roots of $p(x)$ in R and let $b_0 < b_1 < \cdots < b_m$ be the roots of $\widehat{p}(x)$ in $\widehat{R}$. Define $g(a_i) = b_i$ for $i = 0, \ldots, m$. Note that there really must be the same number of roots of $p(x)$ in R as there are roots of $\widehat{p}(x)$ in $\widehat{R}$. This follows from Sturm's Theorem: the number of sign changes in the sequence for $p(x)$ will be the same as the number of sign changes in the sequence for $\widehat{p}(x)$, since f is an isomorphism.

Clearly g is an isomorphism. Moreover, since any isomorphism extending f must send the roots of a polynomial $p(x)$ to roots of $\widehat{p}(x)$, and in the correct order, we see that g is unique. □

We are now ready to prove the main result of this section.

Theorem 3.3. *Any computable ordered field with finite transcendence degree is computably stable.*

Proof. Let F and $\widehat{F}$ be computable ordered fields with finite transcendence degree and $f : F \to \widehat{F}$ an isomorphism. Let R and $\widehat{R}$ be computable copies of the real closures of F and $\widehat{F}$ respectively such that the embeddings $\psi : F \hookrightarrow R$ and $\widehat{\psi} : \widehat{F} \hookrightarrow \widehat{R}$ are computable. By Lemma 3.2, there is an isomorphism $g : R \to \widehat{R}$ which extends f. By Lemma 3.1, we know that g is in fact a computable isomorphism.

To compute $f(t)$ for $t \in F$, we simply find $\psi(t)$, and then $g(\psi(t))$. Since g extended f, we know that $g(\psi(t)) = \widehat{\psi}(f(t))$. But now we can just search through $\widehat{F}$ to find an element $\widehat{t}$ such that $\widehat{\psi}\left(\widehat{t}\right) = g(\psi(t))$. Thus we can compute $f(t)$ for any $t \in F$, so f is a computable isomorphism. □

Realizing that every element of the field can be defined with a formula using a finite number of parameters (the transcendence basis), leads to a proof of a related result. We will show that any computable ordered field with finite transcendence degree is relatively computably categorical.

Definition 3.4. A computable structure $\mathcal{A}$ is *relatively computably categorical* if for every structure $\mathcal{B}$ which is classically isomorphic to $\mathcal{A}$, there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ which is computable from $\mathcal{B}$.

To prove the result, we will appeal to a theorem of Ash, Knight, Manasse, and Slaman [1], and independently Chisholm [2]. We need only the simplest case of the theorem.

Theorem 3.5 (Ash-Knight-Manasse-Slaman, Chisholm)**.** *A structure $\mathcal{A}$ is relatively computably categorical if and only if it has a Σ^0_1 Scott family.*

A structure $\mathcal{A}$ has a Σ^0_1 Scott family if there is a finite sequence $\bar{a} \in \mathcal{A}$ and a Σ^0_1 family of existential formulas $\varphi_i(x, \bar{a})$ such that

(1) Every $b \in \mathcal{A}$ satisfies $\varphi_i(x, \bar{a})$ for at least one i.
(2) If two elements $b, c \in \mathcal{A}$ satisfy the same φ_i, then there is an automorphism of F taking $b \mapsto c$ which fixes $\bar{a}$.

Lemma 3.6. *Let F be a computable ordered field with finite transcendence degree. Then F has a Σ^0_1 Scott family.*

Proof. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1}\rangle$ be a transcendence basis for F. Let $E = \mathbb{Q}(a_0, \ldots a_n) \subseteq F$. We now enumerate a family of formulas $\varphi_{i,j}$ as follows. For each polynomial $p_i \in E[x]$, and each $j \leq k$, we let $\varphi_{i,j}(x, \bar{a})$ be the formula which says that p_i has k roots and x is the jth-least of these k roots. Here k is the actual number of roots of p_i (which can be found computably, using Sturm's Theorem, for example). Since we are allowed parameters $\bar{a}$ in the formula, such $\varphi_{i,j}$ exist for all i and all $j \leq \deg(p_i)$.

We claim that the family of all such $\varphi_{i,j}$ is a Σ^0_1 Scott family for F. First, note that the collection is clearly Σ^0_1, since we provided an effective enumeration of the formulas (the polynomials p_i can be effectively enumerated). Also, the formulas are all existential. Now for any $b \in F$, b is the root of some polynomial $p(x) \in E[x]$, and that polynomial is p_i for some i. Further, there must be some number j of roots of $p(x)$ less than b, so b satisfies $\varphi_{i,j}$. Thus condition (1) is satisfied. Condition (2) is satisfied trivially, since for every $\varphi_{i,j}$, there is no more than one $b \in F$ which satisfies $\varphi_{i,j}$. Therefore $\{\varphi_{i,j}\}$ is a Σ^0_1 Scott family for F. □

Combining Lemma 3.6 with Theorem 3.5, we immediately arrive at:

Theorem 3.7. *Let F be a computable ordered field with finite transcendence degree. Then F is relatively computably categorical.*

Before leaving the finite transcendence degree case, it is worth pointing out that these results relied heavily on the fact that our fields are ordered. Indeed, there are computable algebraic fields which are not computably categorical (see [17]).

4. Ordered Fields with Infinite Transcendence Degree

For computable real closed fields, if the field has infinite transcendence degree, then the computable dimension is infinite. We wish to extend this result to the larger class of computable ordered fields with infinite transcendence degree. Real closed fields are the largest algebraic extension of an ordered field. We now consider the other extreme: purely transcendental fields. Use p_i to denote the ith prime.

Example 4.1. The field $\mathbb{Q}(e^{\sqrt{p_i}})_{i\in\mathbb{N}}$, under the standard ordering, has computable dimension ω.

We will verify this example below, but note first that the field really is a computable ordered field with infinite transcendence degree. The transcendence degree is infinite by the Lindermann-Weierstrass Theorem, which guarantees that $\{e^{\sqrt{p_i}} \mid i \in \mathbb{N}\}$ is algebraically independent since $\{\sqrt{p_i} \mid i \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$. Also notice that the field is archimedean (it is a subfield of $\mathbb{R}$) and is a purely transcendental extension of $\mathbb{Q}$. Finally, there is a computable copy of the field in which $\{e^{\sqrt{p_i}} \mid i \in \mathbb{N}\}$ is computable. This is accomplished by using formal power series to approximate the inequalities for the transcendence basis. (A detailed discussion of this field can be found in [13].) Thus the field is a computable, archimedean, purely transcendental extension of $\mathbb{Q}$ with infinite transcendence degree and a computable pure transcendence basis. We show all such fields have computable dimension ω.

Theorem 4.2. *Let F be a computable archimedean field which is a purely transcendental extension of $\mathbb{Q}$ with infinite transcendence degree, for which there is a computable pure transcendence basis $\mathcal{B}$. Then the computable dimension of F is ω.*

The proof will consists of constructing a field $\widehat{F}$ which is Δ^0_2-isomorphic, but not computably isomorphic, to F. This will be enough, by a theorem of Goncharov.

Theorem 4.3 (Goncharov [7])**.** *If a countable structure $\mathcal{A}$ has two computable copies $\mathcal{A}_1$ and $\mathcal{A}_2$ which are Δ^0_2 isomorphic but not computably isomorphic, then the computable dimension of $\mathcal{A}$ is ω.*

Our proof rests on our ability to redefine a Δ^0_2 isomorphism as we construct it. At any specific stage of the construction, some finite subset A of $\widehat{F}$ has been mentioned, and some of these elements $B = \{b_0, b_1, \dots, b_n\}$ will be intended to be the transcendence basis. The other elements of A will either be rationals or be defined in terms of the elements in B. For example, one element a_7 might be defined to be $(b_0 + 3b_1) \cdot b_2^{-1}$. Additionally, we will have already specified the order on the elements of A. We cannot change the order, or the algebraic relationships between the elements of A. To redefine the isomorphism, we convert an element of B to a rational. Say we want to make b_1 rational. Since a_7 is defined in terms of b_1, we must also change a_7 by picking a rational "close enough" to b_1 so that a_7 (and all the other elements defined in terms of b_1) remain in the same order among all elements of A. That this is possible is a purely algebraic result which we prove as a separate lemma. In what follows we use the following piece of notation: given a quotient of polynomials $p(\bar{x})$ over F and a tuple $\bar{b}$ of the same length as $\bar{x}$, with b_i an element of the tuple $\bar{b}$, by $p(\bar{b})^{b_i}_c$ we will mean the result of replacing all occurrences of b_i in $p(\bar{b})$ with c. In other words, $p(\bar{b})^{b_i}_c = p(\bar{b}')$ where $\bar{b}' = \langle b_0, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n \rangle$.

Lemma 4.4. *Let A be any finite subset of an archimedean field F and let $B = \{b_0, b_1, \dots, b_n\} \subseteq A$. Suppose that for each $a \in A$, there is a quotient of rational polynomials $p_a(\bar{x})$ containing at most the variables $x_0, \dots, x_n$ such that $a = p_a(\bar{b})$ where $\bar{b} = \langle b_0, b_1, \dots, b_n \rangle$. Then for each $b_i \in B$, there is a rational c close enough to b_i so that for all $a, a' \in A$,*

$$p_a(\bar{b}) < p_{a'}(\bar{b}) \textit{ if and only if } p_a(\bar{b})^{b_i}_c < p_{a'}(\bar{b})^{b_i}_c.$$

Proof. Let A and B be as in the statement of the lemma. Fix $b_i \in B$. For each $a \in A$, consider the function $f_a(x) = p_a(\bar{b})^{b_i}_x$. This is simply the quotient of two polynomials in a single variable x with coefficients from $\mathbb{Q}(B)$. As such, there is some $E \subseteq F$ containing b_i on which $f_a : E \to F$ is a continuous function. Now for any $a \in A$, let a_1 and a_2 be such that $a_1 < a < a_2$ and a is the only element of A between a_1 and a_2. Consider the interval

$$I_a = \left(\frac{a_1 + a}{2}, \frac{a + a_2}{2} \right) \cap f_a(E).$$

Now I_a is an open set (in the subspace topology on $f_a(E)$), and since f_a is continuous, $f_a^{-1}(I_a)$ is open and contains b_i. Similarly for each of the finitely many $a \in A$. Let

$$I = \bigcap_{a \in A} f_a^{-1}(I_a).$$

This is the intersection of finitely many open sets, so open. Also, I contains b_i, so I must contain an interval about b_i. Take c to be any rational in this interval.

This c so selected satisfies the lemma. To see this, note that by the choice of c, we have $f_a(c) \in I_a$. Since $f_a(c) = p_a(\bar{b})^{b_i}_c$, it follows that

$$p_a(\bar{b}) < p_{a'}(\bar{b}) \text{ if and only if } p_a(\bar{b})^{b_i}_c < p_{a'}(\bar{b})^{b_i}_c$$

for all a and a' in A. □

We are now ready to prove our theorem.

Proof of Theorem 4.2. Let $F = \{a_0, a_1, \ldots\}$ with a pure transcendence basis $\{a_{i_0}, a_{i_1}, \ldots\}$. We build a computable copy $\widehat{F} = \{b_0, b_1, \ldots\}$ of F along with a Δ^0_2 isomorphism $f : \widehat{F} \to F$. The construction runs in stages, so that by the end of stage s we will have defined $\widehat{F}_s \subset \widehat{F}$ and $f_s : \widehat{F}_s \to F$. We take $\widehat{F} = \bigcup_s \widehat{F}_s$ and $f = \lim_s f_s$. Through the construction, we satisfy the following requirements for all i, e, and s:

P_i: $\lim_s f_s(b_i)$ exists.

R_i: $\exists j\,(f(b_j) = a_i)$.

Q_s: For all $x, y, z \in \widehat{F}_s$,
$f_s(x) + f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$,
$f_s(x) \cdot f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z)$, and
$f_s(x) < f_s(y)$ if and only if $f_{s+1}(x) < f_{s+1}(y)$.

D_e: $\varphi_e \neq f^{-1}$.

Satisfying the P_i and R_i requirements ensures that f is a well defined bijection (our construction makes each f_s an injection). We define addition, multiplication, and the order relation on $\widehat{F}$ by $x + y = f^{-1}(f(x) + f(y))$, $x \cdot y = f^{-1}(f(x) \cdot f(y))$, and $x < y$ if and only if $f(x) < f(y)$. Thus f will in fact be an isomorphism.

Satisfying Q_s for each s ensures that addition, multiplication, and the order relation are computable: to decide whether $x < y$, wait until x and y are in $\widehat{F}_s$, then ask whether $f_s(x) < f_s(y)$. This can be answered, since F is a computable ordered field, and we know

$$x < y \iff f(x) < f(y) \iff f_s(x) < f_s(y).$$

Similarly, to find $x+y$ or $x \cdot y$, we just wait until x and y are in $\widehat{F}_s$. Our construction puts $f_s(x) + f_s(y)$ and $f_s(x) \cdot f_s(y)$ in the range of f_{s+1}, so we have

$$x + y = z \iff f(x) + f(y) = f(z) \iff f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$$

$$x \cdot y = z \iff f(x) \cdot f(y) = f(z) \iff f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z).$$

But we can compute $f_{s+1}(x) + f_{s+1}(y)$ and $f_{s+1}(x) \cdot f_{s+1}(y)$ since F is a computable field, and then search through $\widehat{F}_{s+1}$ until we find the element z for which $f_{s+1}(z)$ is the correct sum or product.

Satisfying D_e for each e guarantees $\widehat{F}$ is not computably isomorphic to F. This works because F and $\widehat{F}$ are archimedean, so any isomorphism between them is unique. But f will be that isomorphism, so making $\varphi_e \neq f^{-1}$ for any e says that f^{-1} (and as such f) is not computable.

So meeting all requirements will give us the desired result. Now on to the construction. It will be useful to label each element of $\widehat{F}$ with a quotient of rational polynomials in some finite number of variables $x_0, x_1, \ldots, x_n$. We do this in such a way that if $p_i(\bar{x})$ is the label for b_i, then $f(b_i) = p_i(\bar{a})$ where $\bar{a} = \langle a_{i_0}, a_{i_1}, \ldots, a_{i_n}\rangle$. Since F is a purely transcendental extension of $\mathbb{Q}$, such a labeling is possible. As the construction proceeds, f_s will need to be redefined on some elements, and in doing so, we will change the label of those elements. The labels tell us how to safely redefine f_s.

Construction: Initially, let $F_0 = \{b_0, b_1, b_2\}$ and define f_0 so that $f_0(b_0) = 0_F$, $f_0(b_1) = 1_F$ and $f_0(b_2) = a_{i_0}$. Give b_0, b_1, and b_2 labels 0, 1 and x_0 respectively. For each stage s, first try to meet a requirement D_e:

(1) Check if there is some $e \leq s$ for which $\varphi_{e,s}(a_{i_e})\downarrow = b_j$ and $f_s(b_j) = a_{i_e}$. If there is no such e, let $f_{s+1}(b_i) = f_s(b_i)$ for all $b_i \in \widehat{F}_s$ and go to step 5. If there is such an e, pick the least one and continue to step 2:

(2) Search for and find a rational c not already in the range of f_s close enough to a_{i_e} in the sense of Lemma 4.4. (More precisely, take A and B in the lemma to be the range of f_s and the elements of the pure transcendence basis for F already in the range, respectively. The lemma guarantees that such a c can be found.) Define $f_{s+1}(b_j) = c$ and relabel b_j with simply c.

(3) For each $b_k \in \widehat{F}_s$ with label $p_k(\bar{x})$, define $f_{s+1}(b_k) = p_k(\bar{a}')$, where

$$\bar{a}' = \langle a_{i_0}, \ldots, a_{i_{e-1}}, c, a_{i_{e+1}}, \ldots, a_{i_n} \rangle.$$

Relabel b_k with $p'_k(\bar{x})$, where p'_k is the result of replacing every occurrence of x_e in $p_k(\bar{x})$ with c (so $p'_k(\bar{a}) = p_k(\bar{a}')$).

(4) For each $b_k \in \widehat{F}_s$ such that $f_s(b_k) \neq f_{s+1}(b_k)$, take k' least such that $b_{k'}$ is not already in the domain of f_{s+1} and define $f_{s+1}(b_{k'}) = f_s(b_k)$. Label $b_{k'}$ with $p_k(\bar{x})$ (the old label of b_k.)

Next, define a little more of f:

(5) For each $b_i, b_j \in \widehat{F}_s$, if any of $f_{s+1}(b_i) + f_{s+1}(b_j)$, $f_{s+1}(b_i) \cdot f_{s+1}(b_j)$, $-f_{s+1}(b_i)$, or $f_{s+1}(b_i)^{-1}$ are not already in the range of f_{s+1}, define f_{s+1} on b_k to be that element, where k is least such that b_k is not already in the domain of f_s. Label b_k accordingly (i.e., if we defined $f_{s+1}(b_k)$ to be $f_{s+1}(b_i) + f_{s+1}(b_j)$, then label b_k with $p_i(\bar{x}) + p_j(\bar{x})$, and similarly for the other cases).

(6) For the least k such that b_k is not already in the domain of f_{s+1}, set $f_{s+1}(b_k) = a_{i_{s+1}}$. Label b_k with x_{s+1}.

(7) Let $\widehat{F}_{s+1}$ be the domain of f_{s+1}.

This completes the construction.

Verification: We verify that each requirement is met. The construction actively worked to satisfy the D_e requirements. For each e such that $\varphi_e(a_{i_e})\downarrow$, either $\varphi_e(a_{i_e}) \neq f_s^{-1}(a_{i_e})$, in which case D_e is satisfied, or else we immediately act to satisfy D_e by defining f_{s+1} so that $f_{s+1}(f_s^{-1}(a_{i_e})) \neq a_{i_e}$. The only stage s for which $f_{s+1}^{-1}(a_{i_e}) \neq f_s^{-1}(a_{i_e})$ is one for which we act to meet D_e, so if we ever act to meet D_e, we will succeed and D_e will be satisfied thenceforth.

To see that each P_i is satisfied, consider for which s it happens that $f_s(b_k) \neq f_{s+1}(b_k)$. The only time in the construction when we redefine f is when acting to meet D_e for some e. We define $f_{s+1}(b_k)$ in terms of the label for b_k, but replacing the variable x_e with a rational c. If the label for b_k does not contain x_e, then we have $f_{s+1}(b_k) = f_s(b_k)$. Otherwise $f_{s+1}(b_k) \neq f_s(b_k)$, but we will only have this situation once for each x_e, since we only act to meet D_e once. Since the label for b_k contains only finitely many variables, we will have $f_{s+1}(b_k) \neq f_s(b_k)$ for only finitely many s. Thus P_i is satisfied for all i.

Similarly, each requirement R_i is met. By the construction, for every $a_i \in F$, there is some stage s at which a_i is in the range of f_s. This is because we put all rationals into the range, and all elements of the pure transcendence basis into the range, and then close under the field operations. We must check though that $\lim_s f_s^{-1}(a_i)$ exists for each $a_i \in F$. The only time $f_s^{-1}(a_i) \neq f_{s+1}^{-1}(a_i)$ is when we change f in acting to meet D_e for some e. If $f_s^{-1}(a_i)$ changes, then it must have been that the label for $f_s^{-1}(a_i)$ contains x_e. Since we need only act to meet D_e at most once for each e, and since there are only finitely many e for which x_e occurs in the label for $f_s^{-1}(a_i)$, we see that there are only finitely many stages s for which $f_s^{-1}(a_i) \neq f_{s+1}^{-1}(a_i)$. Thus R_i is satisfied for all i.

Finally, consider requirement Q_s. Fix $x, y, z \in \widehat{F}_s$. Let $p_x(\bar{x})$, $p_y(\bar{x})$, and $p_z(\bar{x})$ be their labels, respectively, at stage s. Now $p_x(\bar{a}) + p_y(\bar{a}) = p_z(\bar{a})$ if and only if $p_x(\bar{a}') + p_y(\bar{a}') = p_z(\bar{a}')$ (since we are simply substituting a rational in for the variable x_e in each term). But by the construction and how we defined our labeling, we have that $f_s(x) = p_x(\bar{a})$, $f_s(y) = p_y(\bar{a})$, and $f_s(z) = p_y(\bar{a})$. Also, since for any k, $p_k(\bar{a}') = p'_k(\bar{a})$, we have that $f_{s+1}(x) = p_x(\bar{a}')$, $f_{s+1}(y) = p_y(\bar{a}')$, and $f_{s+1}(z) = p_z(\bar{a}')$. Thus

$$f_s(x) + f_s(y) = f_s(z) \Longleftrightarrow p_x(\bar{a}) + p_y(\bar{a}) = p_z(\bar{a}) \Longleftrightarrow \\ \Longleftrightarrow p_x(\bar{a}') + p_y(\bar{a}') = p_z(\bar{a}') \Longleftrightarrow f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z).$$

Similarly

$$f_s(x) \cdot f_s(y) = f_s(z) \Longleftrightarrow f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z).$$

That $f_s(x) < f_s(y)$ if and only if $f_{s+1}(x) < f_{s+1}(y)$ follows from Lemma 4.4: we picked c close enough to a_{i_e} precisely so that this would hold. This completes the verification, and the proof. □

The fields we started with above had computable transcendence basis. However, since copies of the field need not be computably isomorphic, we have no guarantee the copies will have a computable transcendence basis. In fact, with a slight modification to the above proof, it is possible to create a copy of the starting field in which *no* transcendence basis is computable: as you build $\widehat{F}$, make sure that any algebraically independent set computes the halting problem. Alternatively, it is be possible to ensure that any infinite c.e. set in $\widehat{F}$ would necessarily be algebraically dependent, so no transcendence basis can even contain an infinite c.e. set. We state these two results as a corollary and leave the details of the proofs to the reader:

Corollary 4.5. *There are computable fields for which every transcendence basis computes the halting problem. There are computable fields for which every transcendence basis is immune.*

It would be desirable to extend theorem 4.2 to ordered fields which are not purely transcendental. This is problematic however. The concern is that if we take an algebraic extension over a purely transcendental field which adds algebraic relations between transcendental elements, then we might be able to define the elements of the pure transcendence basis. For example, it might be that the transcendental element a is the only (or least) element x of the field for which $1 - x^5$ has a fifth root. The above proof would fail because we would be unable to pick a rational c to set $f_{s+1}(c) = f_s(a)$, if the relevant relations were already present in $\widehat{F}_s$.

As it turns out, it *is* possible to algebraically identify the elements of a transcendence basis in specific algebraic extensions. Thus, we can show:

Theorem 4.6. *There is an archimedean field with infinite transcendence degree which is computably categorical.*

We will not give a proof, as this follows almost immediately from work in [18]. Instead we will briefly discuss the general idea. In their paper, Miller and Schoutens use techniques from algebraic geometry to build a (non-ordered) field of infinite transcendence degree which is computably categorical. They start with $\mathbb{Q}(x_i)_{i\in\mathbb{N}}$ and then adjoin elements y_i such that (x_i, y_i) is a solution to the polynomial $X^{q_i} + Y^{q_i} = 1$, where q_i is one of a specific sequence of odd primes they find. By Fermat's Last Theorem, there are no non-trivial solutions to these polynomials in $\mathbb{Q}$, and what Miller and Schoutens add is that there are in fact exactly six non-trivial solutions to the polynomials in the field they build.

All that we must contribute is that it is possible to build an *ordered* field in the same way. We start with any computable purely transcendental ordered field with infinite transcendence degree and a computable pure transcendence basis: $\mathbb{Q}(x_i)_{i\in\mathbb{N}}$. Then let $y_i = (1 - x_i^{q_i})^{1/q_i}$, using formal power series to approximate the order relations if needed. To see that the field is computably categorical, note that all we must do is first find the image of the transcendence basis $\{x_i \mid i \in \mathbb{N}\}$, and then extend the isomorphism to the rest of the field as in Theorem 3.3. To find the image of the transcendence basis, say $f(x_i)$, we search through F and $\widehat{F}$ to find all six solutions to the Fermat curve $X^{q_i} + Y^{q_i} = 1$. One of these solutions in F will be (x_i, y_i). We use the order relation to determine how many of the five other solutions in F have first component less than x_i. We then know that $(f(x_i), f(y_i))$ must be that solution in $\widehat{F}$ for which there are the same number of solutions in $\widehat{F}$ with first component less than $f(x_i)$. Thus we can determine the isomorphism for the transcendence basis, and then extend it to the entire field.

Note this construction can be used to create both archimedean and non-archimedean ordered fields with infinite transcendence degree but finite computable dimension.

5. Archimedean Fields

We have seen that computable ordered fields with finite transcendence degree have computable dimension 1, while at least some computable ordered fields with infinite transcendence degree have computable dimension ω. But are these the only possibilities or are there any computable ordered fields with finite computable dimension greater than 1? We answer this question for archimedean fields.

Theorem 5.1. *Let F be a computable archimedean field. Then F is Δ^0_2 categorical.*

Proof. Let $f : F \to \widehat{F}$ be the unique isomorphism from F to $\widehat{F}$. Since F is archimedean, every element of F is uniquely determined by the set of rationals below it. Since f is an isomorphism, for any $x \in F$ and rational a, we have $f(a) < f(x)$ if and only if $a < x$. However, for every rational a, we can computably determine $f(a)$. Thus given $x \in F$ and $y \in \widehat{F}$, we can computably determine the truth of $a < x \leftrightarrow f(a) < y$, for any rational a. We have

$$f(x) = y \iff \forall a(a \in \mathbb{Q} \to (a < x \leftrightarrow f(a) < y)) \iff$$
$$\iff \forall a(a \notin \mathbb{Q} \vee (a < x \leftrightarrow f(a) < y)).$$

Since $a \notin \mathbb{Q}$ is Π^0_1, we see that $f(x) = y$ is Π^0_1, so certainly Δ^0_2. □

Corollary 5.2. *If F is a computable archimedean field, then the computable dimension of F is either 1 or ω.*

Proof. Let F be a computable archimedean field. If every computable copy of F is computably isomorphic to F, then F has computable dimension 1. If not, then there is a computable ordered field $\widehat{F}$ which is classically but not computably isomorphic to F. By Theorem 5.1, the unique isomorphism from F to $\widehat{F}$ is Δ^0_2. Thus there are two copies of F, namely F and $\widehat{F}$, which are Δ^0_2 isomorphic but not computably isomorphic, so by Goncharov's Theorem (4.3), the computable dimension of F is ω. □

Note that by Theorem 4.6, the split between computable dimension 1 and ω is not given by the split between finite transcendence degree and infinite transcendence degree, as one might expect. Where exactly the split occurs is an open question.

6. Questions

It is unknown whether there is a (nice) algebraic criterion on computable ordered fields which determines the computable dimension. Ordered fields with finite transcendence degree must be computably categorical, so here ordered fields are easier to analyze than fields in general. It appears though that the infinite transcendence degree case is no easier for ordered fields than non-ordered fields. However, whether the algebraic criterion (whatever it might be) would be identical in the ordered and non-ordered cases is open.

The example given in [17] of a field with finite transcendence degree which is not computably categorical is a non-real field. Therefore we ask whether there is a formally real field (without an order specified) with finite transcendence degree which is not computably categorical.

Although archimedean fields must have computable dimension either 1 or ω, the same is not known for ordered fields in general. Indeed, for non-ordered fields, recent work of Miller, Park, Poonen, Schoutens, and Shlapentokh suggests that there are fields with each finite computable dimension. It is possible to build a non-archimedean ordered field with infinite transcendence degree which is computably categorical (as in Theorem 4.6). However, attempting a construction as we used in Theorem 4.2 to show a non-archimedean field has computable dimension ω yields problems beyond the possibility of transcendental elements being algebraically identifiable. Even if the isomorphism built between F and $\widehat{F}$ is not computable, there may be another isomorphism which is – unlike in the archimedean case, there will be many isomorphisms between the fields.

In Theorem 4.2 we required that the field possess a (copy with a) *computable* pure transcendence basis. It would be nice to eliminate this non-algebraic restriction. It is unclear whether or not this can be done. The proof still goes through if the pure transcendence basis is Π^0_1, however while all computable ordered fields contain a Π^0_1 transcendence basis, it is not clear that the same can be concluded for *pure* transcendence basis. This leads us to a question about the complexity of transcendence bases: are there computable purely transcendental ordered fields in which every pure transcendence basis in every computable copy has complexity greater than Π^0_1?

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