Computable Dimension for Ordered Fields

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Abstract

We study computable dimension for various classes of computable ordered fields. We show that computable ordered fields with finite transcendence degree are computably stable. We then build computable ordered fields of infinite transcendence degree which have infinite computable dimension, but also computably categorical fields with infinite transcendence degree. Additionally, we use similar techniques to build computable ordered fields which possess transcendence bases which are complicated in two senses: every transcendence basis can compute the halting problem, or alternatively, every transcendence basis is immune.

Keywords: computable dimension, computable ordered fields, computably categorical ordered fields, effective algebra, non-computable transcendence bases

2000 MSC: 03D45, 03C57, 12J15

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1. Introduction

Whenever studying the computable content of an algebraic structure, the first step is to present the structures in question in a computable way. This is usually done by coding the elements of the structure's domain by the natural numbers, and ensuring that the functions and relations of the structure are computable functions. Those structures which admit such a coding are the computable ones. But for all of these, there are multiple ways in which we could have represented the elements of the domain. A fundamental question is whether our choice in this representation is important - might picking different computable copies of the structure give us different computable-theoretic results about the structure? For example, if, as in this paper, the structure is an ordered field, might one computable copy of the field have a computable transcendence basis, while another computable copy have only non-computable transcendence bases?

One way to know whether we need worry about this is to know the computable dimension of the structure.

Definition 1.1. The *computable dimension* of a computable structure $\mathcal{A}$ is the number of distinct computable copies (presentations) of the structure, up to computable isomorphism.

Definition 1.2. A computable structure $\mathcal{A}$ is *computably categorical* if every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. That is, if the computable dimension of $\mathcal{A}$ is 1.

Slightly stronger than computably categorical is computably stable:

Definition 1.3. A computable structure $\mathcal{A}$ is *computably stable* if for every computable structure $\mathcal{B}$ classically isomorphic to $\mathcal{A}$, every isomorphism $f : \mathcal{A} \to \mathcal{B}$ is in fact a computable isomorphism.

If a structure is computably categorical, then any computable-theoretic property of one computable copy of the structure must be held by *all* computable copies of the structure. That is, the particular coding of a computably categorical structure does not matter. On the other hand, if there are computable copies of the structure which are, while necessarily isomorphic, not isomorphic by a computable isomorphism, then the coding can be very important.

For a wide variety of algebraic structures, the computable dimension is known. One notable exception is when the structure is a field, where the question of computable dimension has been particularly difficult to answer. It is known for algebraically closed fields, as well as real closed fields, that the field is computably categorical if and only if the transcendence degree of the field is finite. Further, if the transcendence degree is infinite, then the computable dimension is infinite (see [10] and [13]). Little is known beyond these two simplest of examples. There are fields with infinite transcendence degree which are computably categorical (see [12]), and fields with finite transcendence degree which are not (see [3] or [11]). It is unknown whether there are any fields with finite computable dimension greater than 1.

The purpose of this paper is to explore whether progress might be made by considering *ordered* fields. After reviewing some preliminaries in section 2, we show in section 3 that all computable ordered fields with finite transcendence degree are computably categorical (in fact, computably stable). In section 4 we consider computable dimension when the transcendence degree of the ordered field is infinite. We show that in the simplest case the computable dimension is indeed ω, but that there are examples of computably categorical ordered fields as well. While we stop short of determining an algebraic criterion for the computable dimension of ordered fields with infinite transcendence degree, we are able to show that for archimedian fields, the computable dimension cannot be anything besides 1 or ω. This is done in section 5. In section 6 we see an example of what can happen if a field is not computably categorical, by considering how complicated an infinite transcendence basis can be. We take computable ordered field with a computable pure transcendence basis and build a computable copy in which the transcendence basis computes the halting problem. Another computable copy is built for which every transcendence basis is immune. We conclude in section 7 with some remaining questions and ideas for further research.

2. Preliminaries

Before considering any computability theory, let us quickly review some classical definitions and results from the theory of ordered fields. For a more comprehensive introduction, see the chapter 6 in [6], chapter 11 in [7], or [14]. Throughout the paper, all fields have characteristic 0 (as all ordered fields do) and are countable (as all computable fields are).

Definition 2.1. Let F be a field. An *ordering* on F is a linear order $\leq$ (i.e., a total, transitive, antisymmetric binary relation) such that for all $a, b, c \in F$,

1. $a \leq b \Longrightarrow a + c \leq b + c$, and

2. $a \leq b,\ 0 \leq c \Longrightarrow ac \leq bc$.

F is *orderable* if there exists an ordering $\leq$ on F. An *ordered field* is a pair $(F, \leq)$.

Related to ordered fields are real fields.

Definition 2.2. A field F is *formally real* (or simply *real*) provided -1 is not a sum of squares in F.

Classically, a field is orderable if and only if it is formally real. However, this does not hold effectively as there are computable real fields which have no computable ordering. In fact, given any Π^0_1 class C there is a computable real field for which the space of orderings is in Turing degree preserving bijection with C (see [10]).

Given a formally real field F, the algebraic closure of F is no longer real since $x^2 + 1 = 0$ has a root in the algebraic closure, making -1 a square. If we consider a maximal algebraic extension of a real field which is still real, we get a real closure.

Definition 2.3. A field F is *real closed* if F is formally real and no algebraic extension of F is formally real. A *real closure* R_F of a field F is a real closed field which is algebraic over F.

Real closed fields have a unique ordering: the positive elements are simply those which have square roots in the field. This is enough to determine the ordering on the field, as $a < b$ if and only if $b - a$ is positive. Every formally real field has a real closure, although it need not be unique. This is because a given formally real field may admit multiple orderings. For example, $\mathbb{Q}(\sqrt{2})$

is formally real with two orderings: one in which $0 < \sqrt{2}$ and the other in which $\sqrt{2} < 0$. However, if we consider a formally real field and specify the order (that is, consider an ordered field) then the unique order on the real closed field must extend the order on the base field. Thus uniqueness of the real closure is guaranteed:

Theorem 2.4 (Artin-Schreier)**.** *Any ordered field $(F, \leq)$ has a unique (up to isomorphism) real closure.*

Beyond their unique orderings, real closed fields are nice for a variety of reasons. In a real closed field R, every polynomial of odd degree with coefficients in R has a root in R. Also, $R(\sqrt{-1})$ is necessarily the algebraic closure of R. Real closed fields are also nice from a model theory point of view: the theory of real closed fields (in the language of ordered rings) is a complete, decidable theory. This implies that any two real closed field are elementarily equivalent. Since the real numbers, as an ordered field, are a real closed field, this says that any real closed field shares all the first order algebraic and order-theoretic properties of $\mathbb{R}$ (this is called the "Tarski-Principle").

Another nice property we will make heavy use of is that it is possible to determine the number of roots of a given polynomial in a real closed field. There are multiple ways to do this. One way is to use the fact that the theory of real closed fields is complete and decidable. Alternatively, we can appeal to the purely algebraic Sturm's Theorem, which we now discuss in more detail.

Theorem 2.5 (Sturm's Theorem)**.** *Let $p(x)$ be any polynomial with coeffi-*

cients in a real closed field R. Then there is a sequence of polynomials

$$p_0(x), p_1(x), \ldots, p_n(x)$$

such that if $p(\alpha) \neq 0$ and $p(\beta) \neq 0$, then the number of distinct roots of $p(x)$ in the interval $[\alpha, \beta]$ is $V_\alpha - V_\beta$, where V_γ denotes the number of variations in sign of $\{p_0(\gamma), p_1(\gamma), \ldots, p_n(\gamma)\}$.

The polynomials $p_0(x), p_1(x), \ldots, p_n(x)$ can be found effectively. In fact, $p_0(x) = p(x)$, $p_1(x) = p'(x)$ and for $i \geq 2$, $p_i(x)$ is the negative remainder after dividing $p_{i-1}(x)$ by $p_{i-2}(x)$. Since we will be concerned with computable real closed fields, we will be able to calculate $p_i(\gamma)$ for any γ in R and $i = 0, \ldots, n$. Thus we will be able to effectively find V_γ, and as such, the number of roots of $p(x)$ between any α and β which are not roots of $p(x)$. What's more, there is a bound (due to Cauchy) on the roots of a given polynomial, so this allows us to effectively determine the total number of roots of a given polynomial. (For a detailed discussion of Sturm's Theorem, and its proof, see [6].)

The real closure of a field is an algebraic extension, but we will also consider field extensions which are not algebraic. Recall that for any field F (ordered or otherwise) a set $S \subseteq F$ is *algebraically dependent* if for some $n \in \mathbb{N}$ there is a nonzero polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ and distinct $s_1, \ldots, s_n \in S$ such that $p(s_1, \ldots, s_n) = 0$. S is *algebraically independent* if it is not algebraically dependent. A maximal algebraically independent set in F is called a *transcendence basis* for F over $\mathbb{Q}$. The *transcendence degree* of a field F is the cardinality of some transcendence basis for F. Every non-algebraic extension field of $\mathbb{Q}$ has a transcendence basis over $\mathbb{Q}$, and all transcendence bases of a given field have the same cardinality, so these are well defined (see [5]). One can also prove that for any field F, if F is an extension of $\mathbb{Q}$ and has a transcendence basis S, then F is algebraic over the field $\mathbb{Q}(S)$. The field $\mathbb{Q}(S)$ is a *purely transcendental* extension of $\mathbb{Q}$, with a *pure transcendence basis* S. Note that every purely transcendental extension has a pure transcendence basis, but also has transcendence bases which are not pure. (All of this also works for extensions of arbitrary fields instead of $\mathbb{Q}$, but we will only need to consider this simplest of cases.)

Finally, we consider the possibility of infinite elements in an ordered field.

Definition 2.6. For any element a in an ordered field F, define the *absolute value of* a by

$$|a| = \begin{cases} a & \text{if } 0 \leq a \\ -a & \text{if } a < 0 \end{cases}$$

Definition 2.7. An ordered field F is *archimedean* if for all $a \in F$ there is some $n \in \mathbb{N}$ such that $|a| \leq n$.

Now we turn to computability theory. We assume familiarity with the basic ideas from the subject (otherwise, see [15]). Intuitively, an ordered field will be computable if the operations $+$ and $\cdot$ are computable, and the relation $\leq$ is computable. While this is often enough for us, we can also be more precise. We work in the language of ordered rings, so a field F will have a domain $|F|$ and there will be binary function symbols $+_F$ and $\cdot_F$, a binary relation $\leq_F$, and distinguished elements 0_F and 1_F. For F to be a *computable* ordered field, $|F|$ will be a computable subset of $\mathbb{N}$, with $+_F$ and $\cdot_F$ partial computable functions from $|F| \times |F|$ to $|F|$, and $\leq_F \subseteq |F| \times |F|$ a computable relation. Additionally, we are given the elements

0_F and 1_F computably, although these can always be found uniformly by searching through the elements of the field. Of course we want F to be an ordered field, so the usual ordered field axioms must be satisfied. Note that since the domain of F is a subset of $\mathbb{N}$, computable ordered fields (and in general all computable algebraic structures) are necessarily countable.

Elements of fields have both additive and multiplicative inverses, and our definition would not be very good if subtraction and division were not computable functions as well. But these are computable in any computably ordered field. To compute a^{-1} in F, we can simply search through all elements of $|F|$ until we find some b so that $a \cdot b = 1_F$. Since F is a field, such a b will eventually be found, and we can set $b = a^{-1}$. Similarly for $-a$.

Since ordered fields necessarily have characteristic 0, the rationals $\mathbb{Q}$ are contained in every ordered field. If F is a computable ordered field, we can locate any given rational q. That is, we can determine which element of the domain of F corresponds to q. If $q = a/b$ where a and b are integers, we simply compute the sum of 1_F with itself a times, and divide that by the sum of 1_F with itself b times. However, while it is possible to find any rational in F, the subfield of F isomorphic to $\mathbb{Q}$ need not be computable. Given an element $a \in F$, there may be no algorithm which determines whether or not a is rational. The rationals do form a computably enumerable (c.e.) subset of F, but there is no way to know when an element is *not* a rational, unless the field were presented in a particularly nice way.

3. Ordered Fields with Finite Transcendence Degree

We will show that every computable ordered field with finite transcendence degree is computably stable. We begin by verifying the same result for computable real closed fields.

Lemma 3.1. *Any computable real closed field with finite transcendence degree is computably stable.*

Proof. Let $R = \{a_0, a_1, \ldots\}$ be a computable real closed field with finite transcendence degree. Let $\widehat{R}$ be a computable real closed field isomorphic to R via the (classical) isomorphism f. Without loss of generality, assume $\{a_0, \ldots, a_{k-1}\}$ is a transcendence basis for R and that a_k is the multiplicative identity. Then $\{f(a_0), \ldots, f(a_{k-1})\}$ is a transcendence basis for $\widehat{R}$ and $f(a_k)$ is the multiplicative identity in $\widehat{R}$. Let $E = \mathbb{Q}(a_0, \ldots, a_{k-1}) \subseteq R$. We will show that f is in fact a computable isomorphism.

Note first that we can computably determine $f(t)$ for any $t \in E$. This is possible since we know the finite information $f(a_0), \ldots, f(a_{k-1})$ and $f(a_k)$. Every other element t of E is some arithmetic combination (sum, difference, product, or quotient) of these finitely many elements. Once we find what combination gives us t, we can form that same combination in $\widehat{R}$, using the fact that f is an isomorphism. Now suppose $p(x)$ is a polynomial in $E[x]$, say

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Since $c_0, \ldots, c_n$ are in E, we can effectively find the polynomial

$$\widehat{p}(x) = f(c_0) + f(c_1)x + \cdots + f(c_n)x^n$$

in $f(E)[x]$.

To compute $f(t)$ for $t \in R$, we first search for and find a polynomial $p(x) \in E[x]$ such that $p(t) = 0$. There must be one since R is algebraic over E. Once found, we determine the number of roots of $p(x)$ which lie in R (which is the same as the number of roots of $\widehat{p}(x)$ which lie in $\widehat{R}$). This can be done either by using Sturm's theorem, or the completeness of the theory of real closed fields. Once we know the number of roots of $p(x)$, we simply search through R to find all of them. Using the computable order on R, we find m such that there are exactly m roots of $p(x)$ less than t. Next, we search through $\widehat{R}$ to find all the roots of $\widehat{p}(x)$, and specifically find the root $\widehat{t}$ which is greater than exactly m other roots. Since f is an isomorphism, it must be that $f(t) = \widehat{t}$, which we have now found. $\square$

Every ordered field has a unique (up to isomorphism) real closure. If F is a computable ordered field, then there is a computable presentation R_F of its real closure, and a computable embedding from F to R_F (see [9]). We will use this to prove our result, but we need to know that isomorphisms behave nicely when we pass to real closures. The next lemma is purely algebraic.

Lemma 3.2. *Let F and $\widehat{F}$ be ordered fields and $f : F \to \widehat{F}$ be an isomorphism. Let R and $\widehat{R}$ be real closures of F and $\widehat{F}$ respectively. Then f extends to a unique isomorphism $g : R \to \widehat{R}$.*

Proof. Define g as follows. First, for every $a \in F$, let $g(a) = f(a)$. Now let a be an element of $R \setminus F$. Since R is an algebraic extension of F, there is a polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Say $p(x) = c_0 + c_1x + \cdots + c_nx^n$, and define $\widehat{p}(x) = f(c_0) + f(c_1)x + \cdots + f(c_n)x^n$. Let $a_0 < a_1 < \cdots < a_m$ be the roots of $p(x)$ in R and let $b_0 < b_1 < \cdots < b_m$ be the roots of $\widehat{p}(x)$ in $\widehat{R}$. Define $g(a_i) = b_i$ for $i = 0, \ldots, m$. Note that there really must be the same number of roots of $p(x)$ in R as there are roots of $\widehat{p}(x)$ in $\widehat{R}$. This follows from Sturm's Theorem: the number of sign changes in the sequence for $p(x)$ will be the same as the number of sign changes in the sequence for $\widehat{p}(x)$, since f is an isomorphism.

Clearly g is an isomorphism. Moreover, since any isomorphism extending f must send the roots of a polynomial $p(x)$ to roots of $\widehat{p}(x)$, and in the correct order, we see that g is unique. □

We are now ready to prove the main result of this section.

Theorem 3.3. *Any computable ordered field with finite transcendence degree is computably stable.*

Proof. Let F and $\widehat{F}$ be computable ordered fields with finite transcendence degree such that $f : F \to \widehat{F}$ is an isomorphism. Let R and $\widehat{R}$ be computable copies of the real closures of F and $\widehat{F}$ respectively such that the embeddings $\psi : F \hookrightarrow R$ and $\widehat{\psi} : \widehat{F} \hookrightarrow \widehat{R}$ are computable. By Lemma 3.2, there is an isomorphism $g : R \to \widehat{R}$ which extends f. By Lemma 3.1, we know that g is in fact a computable isomorphism.

To compute $f(t)$ for $t \in F$, we simply find $\psi(t)$, and then $g(\psi(t))$. Since g extended f, we know that $g(\psi(t)) = \widehat{\psi}(f(t))$. But now we can just search through $\widehat{F}$ to find an element $\widehat{t}$ such that $\widehat{\psi}\left(\widehat{t}\right) = g(\psi(t))$. Thus we can compute $f(t)$ for any $t \in F$, so f is a computable isomorphism. Since f was arbitrary, we see that F is computably stable. □

To summarize, Theorem 3.3 goes through because we can (non-uniformly) match up transcendence bases of the two fields, and after that everything is determined. Each element a is defined by a polynomial $p(x)$ of which it is a root, along with the number of roots of $p(x)$ in the real closure that are smaller than a. The polynomial can be searched for, and the number of smaller roots can be determined algebraically. Thus corresponding roots can be effectively matched up, so we can effectively determine the isomorphism.

Realizing that every element of the field can be defined with a formula using a finite number of parameters (the transcendence basis), leads us to a quicker proof of a stronger result. We will show that any computable ordered field with finite transcendence degree is relatively computably categorical.

Definition 3.4. A computable structure $\mathcal{A}$ is *relatively computably categorical* if for every structure $\mathcal{B}$ which is classically isomorphic to $\mathcal{A}$, there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ which is computable from $\mathcal{B}$.

To prove the result, we will appeal to a theorem of Ash, Knight, Manasse, and Slaman, and independently Chisholm (see [1] or [2]). We need only the simplest case of the theorem.

Theorem 3.5 (Ash-Knight-Manasse-Slaman, Chisholm)**.** *A structure $\mathcal{A}$ is relatively computably categorical if and only if it has a Σ^0_1 Scott family.*

A structure $\mathcal{A}$ has a Σ^0_1 Scott family if there is a finite sequence $\bar{a} \in \mathcal{A}$ and a Σ^0_1 family of existential formulas $\varphi_i(x, \bar{a})$ such that

1. Every $b \in \mathcal{A}$ satisfies $\varphi_i(x, \bar{a})$ for at least one i.

2. If two elements $b, c \in \mathcal{A}$ satisfy the same φ_i, then there is an automorphism of F taking $b \mapsto c$ which fixes $\bar{a}$.

Lemma 3.6. *Let F be a computable ordered field with finite transcendence degree. Then F has a Σ_1^0 Scott family.*

Proof. Let $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ be a transcendence basis for F. Let $E = \mathbb{Q}(a_0, \ldots a_n) \subseteq F$. We now enumerate a family of formulas $\varphi_{i,j}$ as follows. For each polynomial $p_i \in E[x]$, and each $j \leq k$, we let $\varphi_{i,j}(x, \bar{a})$ be the formula which says that p_i has k roots and x is the jth-least of these k roots. Here k is the actual number of roots of p_i (which can be found computably, using Sturm's Theorem, for example). Since we are allowed parameters $\bar{a}$ in the formula, such $\varphi_{i,j}$ exist for all i and all $j \leq \deg(p_i)$.

We claim that the family of all such $\varphi_{i,j}$ is a Σ_1^0 Scott family for F. First, note that the collection is clearly Σ_1^0, since we provided an effective enumeration of the formulas (the polynomials p_i can be effectively enumerated). Also, the formulas are all existential. Now for any $b \in F$, b is the root of some polynomial $p(x) \in E[x]$, and that polynomial is p_i for some i. Further, there must be some number j of roots of $p(x)$ less than b, so b satisfies $\varphi_{i,j}$. Thus condition (1) is satisfied. Condition (2) is satisfied trivially, since for every $\varphi_{i,j}$, there is no more than one $b \in F$ which satisfies $\varphi_{i,j}$. Therefore $\{\varphi_{i,j}\}$ is a Σ_1^0 Scott family for F. $\square$

Combining Lemma 3.6 with Theorem 3.5, we immediately arrive at:

Theorem 3.7. *Let F be a computable ordered field with finite transcendence degree. Then F is relatively computably categorical.*

Before leaving the finite transcendence degree case, it is worth pointing out that these results relied heavily on the fact that our fields are ordered.

Indeed, there are computable algebraic fields which are not computably categorical (see [3] for the original proof, or [11]). When the fields in question are ordered, it is possible to distinguish the roots of a given polynomial. An automorphism of a field must send each root of a polynomial to a root of that same polynomial. When the field is not ordered, “different” roots can be exchanged. But with an order on the field, we are able to distinguish roots, and it is no longer possible to send, for example, the least root of $p(x)$ to the second least root of $p(x)$.

4. Ordered Fields with Infinite Transcendence Degree

For computable real closed fields, if the field has infinite transcendence degree, then the computable dimension is infinite. We wish to extend this result to the larger class of computable ordered fields with infinite transcendence degree. We begin with the simplest example. We use p_i to denote the ith prime.

Example 4.1. The field $\mathbb{Q}(e^{\sqrt{p_i}})_{i\in\mathbb{N}}$, under the standard ordering, has computable dimension ω.

We will verify this example below, but note first that the field really is a computable ordered field with infinite transcendence degree. The transcendence degree is infinite by the Lindermann-Weierstrass Theorem, which guarantees that $\{e^{\sqrt{p_i}} \mid i \in \mathbb{N}\}$ is algebraically independent since $\{\sqrt{p_i} \mid i \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$. Also notice that the field is archimedian (it is a subfield of $\mathbb{R}$) and is a purely transcendental extension of $\mathbb{Q}$. Finally, there is a computable copy of the field in which $\{e^{\sqrt{p_i}} \mid i \in \mathbb{N}\}$ is computable. This is accomplished by using formal power series to approximate the inequalities for the transcendence basis. (A detailed discussion of this field can be found in [8].) Thus the field in Example 4.1 is a computable, archimedian, purely transcendental extension of $\mathbb{Q}$ with infinite transcendence degree and a computable pure transcendence basis. We show that all such fields have computable dimension ω.

Theorem 4.2. *Let F be a computable archimedean field which is a purely transcendental extension of $\mathbb{Q}$ with infinite transcendence degree, for which there is a computable pure transcendence basis $\mathcal{B}$. Then the computable dimension of F is ω.*

Before giving a proof of the theorem, let us say a little about the general approach. What we will actually demonstrate is that given a suitable computable ordered field F, we can construct a copy $\widehat{F}$ of F in such a way that there is no computable isomorphism from $\widehat{F}$ to F. Since F and $\widehat{F}$ need to be classically isomorphic, there must be *some* isomorphism between them. We will construct an isomorphism, but it will be Δ^0_2, and not computable. To ensure that there is no computable isomorphism from F to $\widehat{F}$, we will diagonalize against all partial computable functions. That is, for each partial computable function φ_e, we will wait for $\varphi_e(x)\downarrow= y$ for some particular x, and if y looks like it could be the image of x under a ordered field isomorphism, we will "redefine" y so that φ_e cannot be an isomorphism. Verifying that we can redefine such a y, without losing the computability of $\widehat{F}$ or the fact that $\widehat{F}$ is isomorphic to F, will be the challenging part of the proof.

In our particular case, since we are assuming that F is archimedean, we are helped greatly by the fact that if $f : \widehat{F} \to F$ is an isomorphism, then f

is unique (F embeds into $\mathbb{R}$, so every $x \in F$ is determined by a Dedekind cut). Thus, since we are building f to be a Δ_2^0 isomorphism, to diagonalize against a partial computable φ_e we need only insist that $\varphi_e(x) \neq f^{-1}(x)$ for some $x \in F$. The x we will use for this diagonalization will be an element of the computable pure transcendence basis. Since any two elements of the pure transcendence basis will be algebraically independent, it will be possible to redefine $f^{-1}(x)$ in the case that $\varphi_e(x) = f^{-1}(x)$. Since $F = \mathbb{Q}(\mathcal{B})$ where $\mathcal{B}$ is the computable pure transcendence basis, we will be able to construct $\widehat{F}$ by specifying which elements are in $f^{-1}(\mathcal{B})$, and then closing under the field operations. This would need to be modified if F were not a purely transcendental extension of the rationals.

Successfully constructing such an $\widehat{F}$ will demonstrate that F cannot be computably categorical - i.e., $\operatorname{comp}\dim(F) > 1$. However, since F is isomorphic to $\widehat{F}$ via a Δ_2^0 isomorphism, this is enough to demonstrate that $\operatorname{comp}\dim(F) = \omega$. For this, we appeal to a theorem of Goncharov.

Theorem 4.3 (Goncharov [4]). *If a countable structure $\mathcal{A}$ has two computable copies $\mathcal{A}_1$ and $\mathcal{A}_2$ which are Δ_2^0 isomorphic but not computably isomorphic, then the computable dimension of $\mathcal{A}$ is ω.*

The proof of Theorem 4.2 rests heavily on our ability to redefine the Δ_2^0 isomorphism as we construct it. At any specific stage of the construction, we will have mentioned some finite subset A of the field, and some of these elements $B = \{b_0, b_1, \ldots, b_n\}$ will be intended to be our transcendence basis. The other elements of A will either be rationals or be defined in terms of the elements in B. For example, one element a_7 might be defined to be $(b_0 + 3b_1) \cdot b_2^{-1}$. Additionally, we will have already specified the order on the elements of A. We cannot change the order, or the algebraic relationships between the elements of A. When we redefine our isomorphism, we will need to convert one of the elements of B to a rational. Say we want to make b_1 rational. Since a_7 is defined in terms of b_1, we will in fact also be changing a_7. We must pick a rational close enough to b_1 so that a_7 (and all the other elements defined in terms of b_1) remain in the same order among all elements of A. We will prove that this is always possible. This algebraic result will also be useful in section 6, so we will take care of it as a separate lemma. In what follows we use the following piece of notation: given a quotient of polynomials $p(\bar{x})$ over F and a tuple $\bar{b}$ of the same length as $\bar{x}$, with b_i an element of the tuple $\bar{b}$, by $p(\bar{b})^{b_i}_c$ we will mean the result of replacing all occurrences of b_i in $p(\bar{b})$ with c. In other words, $p(\bar{b})^{b_i}_c = p(\bar{b}')$ where $\bar{b}' = \langle b_0, \ldots, b_{i-1}, c, b_{i+1}, \ldots, b_n\rangle$.

Lemma 4.4. *Let A be any finite subset of an archimedean field F and let $B = \{b_0, b_1, \ldots, b_n\} \subseteq A$. Suppose that for each $a \in A$, there is a quotient of rational polynomials $p_a(\bar{x})$ containing at most the variables $x_0, \ldots, x_n$ such that $a = p_a(\bar{b})$ where $\bar{b} = \langle b_0, b_1, \ldots, b_n\rangle$. Then for each $b_i \in B$, there is a rational c close enough to b_i so that for all $a, a' \in A$,*

$$p_a(\bar{b}) < p_{a'}(\bar{b}) \text{ if and only if } p_a(\bar{b})^{b_i}_c < p_{a'}(\bar{b})^{b_i}_c.$$

Proof. Let A and B be as in the statement of the lemma. Fix $b_i \in B$. For each $a \in A$, consider the function $f_a(x) = p_a(\bar{b})^{b_i}_x$. This is simply the quotient of two polynomials in a single variable x with coefficients from $\mathbb{Q}(B)$. As such, there is some $E \subseteq F$ containing b_i on which $f_a : E \to F$ is a continuous function. Now for any $a \in A$, let a_1 and a_2 be such that $a_1 < a < a_2$ and a

is the only element of A between a_1 and a_2. Consider the interval

$$I_a = \left(\frac{a_1 + a}{2}, \frac{a + a_2}{2} \right) \cap f_a(E).$$

Now I_a is an open set (in the subspace topology on $f_a(E)$), and since f_a is continuous, $f_a^{-1}(I_a)$ is open and contains b_i. Similarly for each of the finitely many $a \in A$. Let

$$I = \bigcap_{a \in A} f_a^{-1}(I_a).$$

This is the intersection of finitely many open sets, so open. Also, I contains b_i, so I must contain an interval about b_i. Take c to be any rational in this interval.

This c so selected satisfies the lemma. To see this, note that by the choice of c, we have $f_a(c) \in I_a$. Since $f_a(c) = p_a(\bar{b})_c^{b_i}$, it follows that

$$p_a(\bar{b}) < p_{a'}(\bar{b}) \text{ if and only if } p_a(\bar{b})_c^{b_i} < p_{a'}(\bar{b})_c^{b_i}$$

for all a and a' in A. $\square$

We are now ready to prove our theorem.

Proof of Theorem 4.2. Let $F = \{a_0, a_1, \ldots\}$ with a pure transcendence basis $\{a_{i_0}, a_{i_1}, \ldots\}$. We will build a computable copy $\widehat{F} = \{b_0, b_1, \ldots\}$ of F along with a Δ_2^0 isomorphism $f : \widehat{F} \to F$. The construction will run in stages, so that by the end of stage s we will have defined $\widehat{F}_s \subset \widehat{F}$ and $f_s : \widehat{F}_s \to F$. We will take $\widehat{F} = \bigcup_s \widehat{F}_s$ and $f = \lim_s f_s$. Through the construction, we will satisfy the following requirements for all i, e, and s:

P_i: $\lim_s f_s(b_i)$ exists.

R_i: $\exists j\,(f(b_j) = a_i)$.

Q_s: For all $x, y, z \in \widehat{F}_s$,

$f_s(x) + f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$,

$f_s(x) \cdot f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z)$, and

$f_s(x) < f_s(y)$ if and only if $f_{s+1}(x) < f_{s+1}(y)$.

D_e: $\varphi_e \neq f^{-1}$.

Satisfying the P_i and R_i requirements will ensure that f is a well defined bijection (our construction will be such that each f_s is an injection). We will define addition, multiplication, and the order relation on $\widehat{F}$ by $x + y = f^{-1}(f(x)+f(y))$, $x \cdot y = f^{-1}(f(x) \cdot f(y))$, and $x < y$ if and only if $f(x) < f(y)$. Thus f will in fact be an isomorphism.

Satisfying Q_s for each s will ensure that addition, multiplication, and the order relation will be computable: to decide whether $x < y$, wait until x and y are in $\widehat{F}_s$, then ask whether $f_s(x) < f_s(y)$. This can be answered, since F is a computable ordered field, and we will know

$$x < y \iff f(x) < f(y) \iff f_s(x) < f_s(y).$$

Similarly, to find $x + y$ or $x \cdot y$, we just wait until x and y are in $\widehat{F}_s$. Our construction will put $f_s(x) + f_s(y)$ and $f_s(x) \cdot f_s(y)$ in the range of f_{s+1}, so we will have

$$x + y = z \iff f(x) + f(y) = f(z) \iff f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$$

$$x \cdot y = z \iff f(x) \cdot f(y) = f(z) \iff f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z).$$

But we can compute $f_{s+1}(x) + f_{s+1}(y)$ and $f_{s+1}(x) \cdot f_{s+1}(y)$ since F is a computable field, and then search through $\widehat{F}_{s+1}$ until we find the element z for which $f_{s+1}(z)$ is the correct sum or product.

Satisfying D_e for each e will ensure that $\widehat{F}$ is not computably isomorphic to F. This works because F and $\widehat{F}$ are archimedean, so any isomorphism between them is unique. But f will be that isomorphism, so making $\varphi_e \neq f^{-1}$ for any e will say that f^{-1} (and as such f) is not computable.

So meeting all requirements will give us the desired result. Now on to the construction. It will be useful to label each element of $\widehat{F}$ with a quotient of rational polynomials in some finite number of variables $x_0, x_1, \ldots, x_n$. We will do this in such a way that if $p_i(\bar{x})$ is the label for b_i, then $f(b_i) = p_i(\bar{a})$ where $\bar{a} = \langle a_{i_0}, a_{i_1}, \ldots, a_{i_n} \rangle$. Since F is a purely transcendental extension of $\mathbb{Q}$, such a labeling is possible. As the construction proceeds, f_s will need to be redefined on some elements, and in doing so, we will change the label of those elements. The labels will tell us how to safely redefine f_s.

<u>Construction:</u> Initially, let $F_0 = \{b_0, b_1, b_2\}$ and define f_0 so that $f_0(b_0) = 0_F$, $f_0(b_1) = 1_F$ and $f_0(b_2) = a_{i_0}$. Give b_0, b_1, and b_2 labels 0, 1 and x_0 respectively. For each stage s, first try to meet a requirement D_e:

1. Check if there is some $e \leq s$ for which $\varphi_{e,s}(a_{i_e})\downarrow = b_j$ and $f_s(b_j) = a_{i_e}$. If there is no such e, let $f_{s+1}(b_i) = f_s(b_i)$ for all $b_i \in \widehat{F}_s$ and go to step 5. If there is such an e, pick the least one and continue to step 2:

2. Search for and find a rational c not already in the range of f_s close enough to a_{i_e} in the sense of Lemma 4.4. (More precisely, we take A and B in the lemma to be the range of f_s and the elements of the

pure transcendence basis for F already in the range, respectively. The lemma guarantees that such a c can be found.) Define $f_{s+1}(b_j) = c$ and relabel b_j with simply c.

3. For each $b_k \in \widehat{F}_s$ with label $p_k(\bar{x})$, define $f_{s+1}(b_k) = p_k(\bar{a}')$, where

$$\bar{a}' = \langle a_{i_0}, \ldots, a_{i_{e-1}}, c, a_{i_{e+1}}, \ldots, a_{i_n} \rangle.$$

   Relabel b_k with $p'_k(\bar{x})$, where p'_k is the result of replacing every occurrence of x_e in $p_k(\bar{x})$ with c (so $p'_k(\bar{a}) = p_k(\bar{a}')$).

4. For each $b_k \in \widehat{F}_s$ such that $f_s(b_k) \neq f_{s+1}(b_k)$, take k' least such that $b_{k'}$ is not already in the domain of f_{s+1} and define $f_{s+1}(b_{k'}) = f_s(b_k)$. Label $b_{k'}$ with $p_k(\bar{x})$ (the old label of b_k.)

Next, define a little more of f:

5. For each $b_i, b_j \in \widehat{F}_s$, if any of $f_{s+1}(b_i) + f_{s+1}(b_j)$, $f_{s+1}(b_i) \cdot f_{s+1}(b_j)$, $-f_{s+1}(b_i)$, or $f_{s+1}(b_i)^{-1}$ are not already in the range of f_{s+1}, define f_{s+1} on b_k to be that element, where k is least such that b_k is not already in the domain of f_s. Label b_k accordingly (i.e., if we defined $f_{s+1}(b_k)$ to be $f_{s+1}(b_i) + f_{s+1}(b_j)$, then label b_k with $p_i(\bar{x}) + p_j(\bar{x})$, and similarly for the other cases).

6. For the least k such that b_k is not already in the domain of f_{s+1}, set $f_{s+1}(b_k) = a_{i_{s+1}}$. Label b_k with x_{s+1}.

7. Let $\widehat{F}_{s+1}$ be the domain of f_{s+1}.

This completes the construction.

Verification: We verify that each requirement is met. The construction actively worked to satisfy the D_e requirements. For each e such that $\varphi_e(a_{i_e})\downarrow$, either $\varphi_e(a_{i_e}) \neq f_s^{-1}(a_{i_e})$, in which case D_e is satisfied, or else we immediately act to satisfy D_e by defining f_{s+1} so that $f_{s+1}(f_s^{-1}(a_{i_e})) \neq a_{i_e}$. The only stage s for which $f_{s+1}^{-1}(a_{i_e}) \neq f_s^{-1}(a_{i_e})$ is one for which we act to meet D_e, so if we ever act to meet D_e, we will succeed and D_e will be satisfied thenceforth.

To see that we satisfy P_i for each i, we consider for which s it happens that $f_s(b_k) \neq f_{s+1}(b_k)$. The only time in the construction when we redefine f is when acting to meet D_e for some e. We define $f_{s+1}(b_k)$ in terms of the label for b_k, but replacing the variable x_e with a rational c. If the label for b_k does not contain x_e, then we have $f_{s+1}(b_k) = f_s(b_k)$. Otherwise $f_{s+1}(b_k) \neq f_s(b_k)$, but we will only have this situation once for each x_e, since we only act to meet D_e once. Since the label for b_k contains only finitely many variables, we will have $f_{s+1}(b_k) \neq f_s(b_k)$ for only finitely many s. Thus P_i is satisfied for all i.

Similarly, each requirement R_i is met. By the construction, for every $a_i \in F$, there is some stage s at which a_i is in the range of f_s. This is because we put all rationals into the range, and all elements of the pure transcendence basis into the range, and then close under the field operations. We must check though that $\lim_s f_s^{-1}(a_i)$ exists for each $a_i \in F$. The only time $f_s^{-1}(a_i) \neq f_{s+1}^{-1}(a_i)$ is when we change f in acting to meet D_e for some e. If $f_s^{-1}(a_i)$ changes, then it must have been that the label for $f_s^{-1}(a_i)$ contains x_e. Since we need only act to meet D_e at most once for each e, and since there are only finitely many e for which x_e occurs in the label for $f_s^{-1}(a_i)$, we see that there are only finitely many stages s for which $f_s^{-1}(a_i) \neq f_{s+1}^{-1}(a_i)$.

Thus R_i is satisfied for all i.

Finally, we consider requirement Q_s. Fix $x, y, z \in \widehat{F}_s$. Let $p_x(\bar{x})$, $p_y(\bar{x})$, and $p_z(\bar{x})$ be their labels, respectively, at stage s. Now $p_x(\bar{a}) + p_y(\bar{a}) = p_z(\bar{a})$ if and only if $p_x(\bar{a}') + p_y(\bar{a}') = p_z(\bar{a}')$ (since we are simply substituting a rational in for the variable x_e in each term). But by the construction and how we defined our labeling, we have that $f_s(x) = p_x(\bar{a})$, $f_s(y) = p_y(\bar{a})$, and $f_s(z) = p_y(\bar{a})$. Also, since for any k, $p_k(\bar{a}') = p'_k(\bar{a})$, we have that $f_{s+1}(x) = p_x(\bar{a}')$, $f_{s+1}(y) = p_y(\bar{a}')$, and $f_{s+1}(z) = p_z(\bar{a}')$. Thus

$$f_s(x) + f_s(y) = f_s(z) \iff p_x(\bar{a}) + p_y(\bar{a}) = p_z(\bar{a}) \iff$$
$$\iff p_x(\bar{a}') + p_y(\bar{a}') = p_z(\bar{a}') \iff f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z).$$

Similarly

$$f_s(x) \cdot f_s(y) = f_s(z) \iff f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z).$$

That $f_s(x) < f_s(y)$ if and only if $f_{s+1}(x) < f_{s+1}(y)$ follows from Lemma 4.4: we picked c close enough to a_{i_e} precisely so that this would hold. This completes the verification, and the proof. □

It would be desirable to extend this result to ordered fields which are not purely transcendental. This is problematic however. The concern is that if we take an algebraic extension over a purely transcendental field which adds algebraic relations between transcendental elements, then we might be able to define the elements of the pure transcendence basis. For example, it might be that the transcendental element a is the only (or least) element x of the field for which $1 - x^5$ has a fifth root. The above proof would fail because we would be unable to pick a rational c to set $f_{s+1}(c) = f_s(a)$, if the relevant relations were already present in $\widehat{F}_s$.

As it turns out, it is possible to algebraically identify the elements of a transcendence basis in specific algebraic extensions. Thus, we can show:

Theorem 4.5. *There is a computably categorical archimedian field with infinite transcendence degree.*

We will not give a proof, as this follows almost immediately from work in [12]. Instead we will briefly discuss the general idea. In their paper, Miller and Schoutens use techniques from algebraic geometry to build a (non-ordered) field of infinite transcendence degree which is computably categorical. They start with $\mathbb{Q}(x_i)_{i\in\mathbb{N}}$ and then adjoin elements y_i such that (x_i, y_i) is a solution to the polynomial $X^{q_i} + Y^{q_i} = 1$, where q_i is one of a specific sequence of odd primes they find. By Fermat's Last Theorem, there are no non-trivial solutions to these polynomials in $\mathbb{Q}$, and what Miller and Schoutens add is that there are in fact exactly six non-trivial solutions to the polynomials in the field they build.

All that we must add is that it is possible to build an *ordered* field in the same way. We start with any computable purely transcendental ordered field with infinite transcendence degree and a computable pure transcendence basis: $\mathbb{Q}(x_i)_{i\in\mathbb{N}}$. For example, we could use the archimedian field in example 4.1, in which case $x_i = e^{\sqrt{p_i}}$. Then let $y_i = (1 - x_i^{q_i})^{1/q_i}$, using formal power series to approximate the order relations if needed. To see that the field is computably categorical, note that all we must do is first find the image of the transcendence basis $\{x_i \mid i \in \mathbb{N}\}$, and then extend the isomorphism to the rest of the field as in Theorem 3.3. To find the image of the transcendence basis, say $f(x_i)$, we search through F and $\widehat{F}$ to find all six solutions to the Fermat curve $X^{q_i} + Y^{q_i} = 1$. One of these solutions in F will be (x_i, y_i). We use the order relation to determine how many of the five other solutions in F have first component less than x_i. We then know that $(f(x_i), f(y_i))$ must be that solution in $\widehat{F}$ for which there are the same number of solutions in $\widehat{F}$ with first component less than $f(x_i)$. Thus we can determine the isomorphism for the transcendence basis, and then extend it to the entire field.

Note that if we start with a computable ordered field (with computable pure transcendence basis) which is not archimedian, then this construction produces a computably categorical non-archimedian ordered field with infinite transcendence degree.

5. Archimedian Fields

We have seen that computable ordered fields with finite transcendence degree have computable dimension 1, while at least some computable ordered fields with infinite transcendence degree have computable dimension ω. But are these the only possibilities? That is, are there any computable ordered fields with finite computable dimension greater than 1? We will see that this is not possible, at least when considering archimedean fields.

Theorem 5.1. *Let F be a computable archimedean field. Then F is Δ^0_2 categorical.*

Proof. Let $f : F \to \widehat{F}$ be the unique isomorphism from F to $\widehat{F}$. Since F is archimedean, every element of F is uniquely determined by the set of rationals below it. Since f is an isomorphism, for any $x \in F$ and rational

a, we have $f(a) < f(x)$ if and only if $a < x$. However, for every rational a, we can computably determine $f(a)$. Thus given $x \in F$ and $y \in \widehat{F}$, we can computably determine the truth of $a < x \leftrightarrow f(a) < y$, for any rational a. We have

$$\begin{aligned} f(x) = y \iff & \forall a(a \in \mathbb{Q} \to (a < x \leftrightarrow f(a) < y)) \\ & \iff \forall a(a \notin \mathbb{Q} \vee (a < x \leftrightarrow f(a) < y)). \end{aligned}$$

Since $a \notin \mathbb{Q}$ is Π^0_1, we see that $f(x) = y$ is Π^0_1, so certainly Δ^0_2. □

Corollary 5.2. *If F is a computable archimedean field, then the computable dimension of F is either 1 or ω.*

Proof. Let F be a computable archimedean field. If every computable copy of F is computably isomorphic to F, then F has computable dimension 1. If not, then there is a computable ordered field $\widehat{F}$ which is classically but not computably isomorphic to F. By Theorem 5.1, the unique isomorphism from F to $\widehat{F}$ is Δ^0_2. Thus there are two copies of F, namely F and $\widehat{F}$, which are Δ^0_2 isomorphic but not computably isomorphic, so by Goncharov's Theorem (4.3), the computable dimension of F is ω. □

Note that by Theorem 4.5, the split between computable dimension 1 and ω is not given by the split between finite transcendence degree and infinite transcendence degree, as one might expect. Where exactly the split occurs is an open question.

6. Transcendence Bases

We will produce a presentation of a computable ordered field which has no computable transcendence basis. In fact, we will be able to do more.

It is possible to construct a computable archimedean field for which every transcendence basis computes the halting problem.

Theorem 6.1. *There is a computable archimedean field for which every transcendence basis computes K.*

The proof is similar to that of 4.2, although there are enough differences that we will give the full proof.

Proof. Let F be a computable archimedean field, purely transcendental over $\mathbb{Q}$, with a computable pure transcendence basis (for instance, as in Example 4.1). For ease of notation, let $F = \{a_0, a_1, \ldots\}$ and let $\{a_{i_0}, a_{i_1}, \ldots\}$ be a computable pure transcendence basis of F. We will build a computable ordered field $\widehat{F}$ and a Δ^0_2 isomorphism $f : \widehat{F} \to F$. We will use $\{b_0, b_1, \ldots\}$ for the elements of $\widehat{F}$. We approximate $\widehat{F}$ with finite $\widehat{F}_s$ at stage s, and approximate f with $f_s : \widehat{F}_s \to F$, so that $\widehat{F} = \bigcup_s \widehat{F}_s$ and $f = \lim_s f_s$. For the construction, we also fix a computable enumeration K_s of K so that $K_{s+1} \setminus K_s$ contains at most one element.

As in the proof of Theorem 4.2, we will satisfy requirements:

P_i: $\lim_s f_s(b_i)$ exists.

R_i: $\exists j(f(b_j) = a_i)$.

Q_s: For all $x, y, z \in \widehat{F}_s$,

$f_s(x) + f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$,

$f_s(x) \cdot f_s(y) = f_s(z)$ if and only if $f_{s+1}(x) \cdot f_{s+1}(y) = f_{s+1}(z)$, and

$f_s(x) < f_s(y)$ if and only if $f_{s+1}(x) < f_{s+1}(y)$.

Doing so will ensure that $\widehat{F}$ is isomorphic to F and that $\widehat{F}$ is computable. Instead of the requirement that $f^{-1} \neq \varphi_e$ for any e, we will have the requirement:

D_n: If $\{b_{i_0}, \ldots, b_{i_{n-1}}\}$ is algebraically independent, then $K\restriction n = K_m\restriction n$ where $m = \max\{i_0, \ldots, i_{n-1}\}$.

If we meet D_n for each n, then any transcendence basis $\{b_{i_0}, b_{i_1}, \ldots\}$ of $\widehat{F}$ will compute K, as follows. To determine whether $x \in K$, consider any x elements of the transcendence basis, say $\{b_{i_0}, \ldots, b_{i_{x-1}}\}$. Then calculate $m = \max\{i_0, \ldots, i_{x-1}\}$ and enumerate K for m stages. Since $\{b_{i_0}, \ldots, b_{i_{x-1}}\}$ is algebraically independent, and D_x was satisfied, we have $K\restriction x = K_m\restriction x$. But K_m is computable, so we can ask whether $x \in K_m$. Whatever the answer, that will be the answer as to whether $x \in K$.

As before, at each stage s of the construction we will label each $b_i \in \widehat{F}_s$ with $p_i(\bar{x})$, a quotient of rational polynomials in variables $\{x_0, x_1, \ldots, x_s\}$. Our labeling will satisfy $f_s(b_i) = p_i(\bar{a})$ where $\bar{a} = \langle a_{i_0}, \ldots, a_{i_s}\rangle$. Such a labeling is possible since F is a purely transcendental extension of $\mathbb{Q}$.

<u>Construction:</u> We initially set $\widehat{F}_0 = \{b_0, b_1, b_2\}$, $f_0(b_0) = 0_F$, $f_0(b_1) = 1_F$, and $f_0(b_2) = a_{i_0}$. We give b_0, b_1, and b_2 the labels 0, 1, and x_0 respectively. At any stage s of the construction, we perform the following steps:

1. Compute K_{s+1}. If $K_{s+1} \setminus K_s = \emptyset$ or if $K_{s+1} \setminus K_s = \{n\}$ for $n > s$, let $f_{s+1}(b_i) = f_s(b_i)$ for all $b_i \in F_s$, and do not change the labels of any of those b_i. Then skip to step 3. Otherwise, proceed to step 2.

2. We have $K_{s+1} \setminus K_s = \{n\}$ with $n \leq s$. We will redefine $f_{s+1}(b_j)$ to

be rational for those b_j such that $f_s(b_j)$ is among $a_{i_n}, a_{i_{n+1}}, \dots a_{i_s}$. Set $m = n$, and $f_{s_m} = f_s$. Then as long as $m \leq s$, do the following:

(a) Let b_j be such that $f_s(b_j) = a_{i_m}$. Search for a rational c_m not already in the range of f_{s_m} close enough to a_{i_m} in the sense of Lemma 4.4. (More precisely, we take A and B in the lemma to be the range of f_{s_m} and the the elements of the pure transcendence basis for F already in the range, respectively. The lemma guarantees that such a c_m can be found.) Set $f_{s_{m+1}}(b_j) = c_m$. Relabel b_j with c_m.

(b) For each $b_k \in \mathrm{dom}(f_{s_m})$ with a label $p_k(\bar{x})$, define $f_{s_{m+1}}(b_k) = p_k(\bar{a}')$, where $\bar{a}'$ is the result of replacing each a_{i_m} in $\bar{a}$ with the rational c_m found above. Relabel b_k with $p'_k(\bar{x})$, where p'_k is the results of replacing every occurrence of x_m in $p_k(\bar{x})$ with c_m (so $p'_k(\bar{a}) = p_k(\bar{a}')$).

(c) For each $b_k \in \mathrm{dom}(f_{s_m})$ such that $f_{s_m}(b_k) \neq f_{s_{m+1}}(b_k)$, take k' least such that $b_{k'}$ is not already in the domain of $f_{s_{m+1}}$ and define $f_{s_{m+1}}(b_{k'}) = f_{s_m}(b_k)$. Label $b_{k'}$ with $p_k(\bar{x})$ (the old label of b_k.)

(d) If $m < s$, increment m and go back to step 2a. If $m = s$, then set $f_{s+1} = f_{s_m}$ and continue to step 3.

Continue as in Theorem 4.2:

3. For each $b_i, b_j \in \widehat{F}_s$, if any of $f_{s+1}(b_i) + f_{s+1}(b_j)$, $f_{s+1}(b_i) \cdot f_{s+1}(b_j)$, $-f_{s+1}(b_i)$, or $f_{s+1}(b_i)^{-1}$ are not already in the range of f_{s+1}, define f_{s+1} on b_k to be that element, where k is least such that b_k is not already

in the domain of f_{s+1}. Label b_k accordingly (i.e., if we defined $f_{s+1}(b_k)$ to be $f_{s+1}(b_i) + f_{s+1}(b_j)$, then label b_k with $p_i(\bar{x}) + p_j(\bar{x})$, and similarly for the other cases).

4. For the least k such that b_k is not already in the domain of f_{s+1}, define $f_{s+1}(b_k) = a_{i_{s+1}}$. Label b_k with x_{s+1}.

5. Let $\widehat{F}_{s+1}$ be the domain of f_{s+1}.

This completes the construction.

Verification: We verify that each requirement is met. Each Q requirement is satisfied by the construction, exactly as in the proof of Theorem 4.2. To see that we satisfy P_i for each i, consider an s for which $f_{s+1}(b_i) \neq f_s(b_i)$. Then we must have reached step 2 in the construction, so some $n < s$ just entered K. Further, the label for b_i must contain an occurrence of x_m for some $n \leq m \leq s$, since if it did not, then $\bar{a}' = \bar{a}$ so

$$f_{s+1}(b_i) = p_i(\bar{a}') = p_i(\bar{a}) = f_s(b_i).$$

After relabeling, the label for b_i will have one or more fewer variables. Thus $f_{s+1}(b_i) \neq f_s(b_i)$ can happen only finitely many times (at most once for each variable in the original label of b_i). So P_i is satisfied.

To see that each requirement R_i is met, note that since 1_F and every element of the transcendence basis is eventually put into the range of f_s and we close the range under the field operations, each a_i is in the range of f_s for some s. We must argue that $\lim_s f_s^{-1}(a_i)$ exists for each i. The label for $f_s^{-1}(a_i)$ mentions only finitely many x_n, so let n be largest so that x_n is in the label for $f_s^{-1}(a_i)$. Let s' be a stage such that $K_{s'} \upharpoonright n = K \upharpoonright n$. Since after stage s', we only change f_s on elements with labels containing x_m for $m > n$, we will have $f_{s'}^{-1}(a_i) = \lim_s f_s^{-1}(a_i)$. So R_i is satisfied.

Finally, let us argue that D_n is satisfied for each n. Suppose for some m, $K\restriction n \neq K_m\restriction n$. Then for some $s > m$, there is an $n' \leq n$ such that $K_{s+1} \setminus K_s = \{n'\}$. Then at stage s of our construction, we relabel all $b_i \in F_s$ so that no label contains variables x_j with $j \geq n'$ (each such variable is replaced with a rational). When we move on to the part of the construction where we pick new elements for F_{s+1} and give them labels containing x_j for $j \geq n'$, we use elements b_k for k not yet used. But the construction is such that the smallest available k is much larger than s. So after stage s of the construction, the only variables occurring in any b_k for $k < s$ are x_j for $j < n'$. So any set of n elements b_k with $k < m < s$ must mention fewer than $n < n'$ variables. Thus such a set of n elements is algebraically dependent (each element is in the algebraic span of a set of fewer than n elements). Therefore D_n is satisfied. This completes the verification, and the proof. $\square$

One may wonder if it is possible to get any stronger results along this line. For instance, is there a computable ordered field such that every transcendence basis is Π^0_2 complete? The answer is no. It is relatively easy to see that every computable ordered field contains a transcendence basis which is Π^0_1:

Proposition 6.2. *Every computable (ordered) field contains a Π^0_1 transcendence basis.*

Proof. Let F be a computable field (or ordered field). Without loss of generality, assume that F has infinite transcendence degree. We will approximate the transcendence basis in stages, having at stage s a set A_s which contains a transcendence basis for F. We will do so in such a way that $\lim_s A_s = \bigcap_s A_s = A$ is a transcendence basis for F. Also, we will ensure that the complement of A is c.e., so the transcendence basis will be co-c.e., that is, Π^0_1.

Fix an enumeration $\{p_0, p_1, \ldots\}$ of all non-zero polynomials in $\mathbb{Q}[x_0, x_1, \ldots, x_n]$ (for all values of n), as well as an enumeration $\{\bar{a}_0, \bar{a}_1, \ldots\}$ of all tuples (of all sizes) of elements from F. Let $A_0 = F$. To form A_s, check whether $p_i(\bar{a}_j) = 0$ for each $i, j \leq s$ with $\bar{a}_j \in A^n_{s-1}$ (for $n = |\bar{a}_j|$). If there is a first $\bar{a}_j = (a_{j_0}, \ldots a_{j_{n-1}})$ for which this is satisfied, let $A_s = A_{s-1} \setminus \{a_{j_{n-1}}\}$. If there is no tuple which makes a polynomial 0, then let $A_s = A_{s-1}$. Now $A = \lim_s A_s$ is clearly a transcendence base for F, and its complement is c.e., as required. □

Above we built a computable archimedean field for which every transcendence basis computed the halting problem. As such, no transcendence basis of that field is computable. Moreover, since the field contains a co-c.e. transcendence basis, there must be a transcendence basis which is not c.e. We now consider whether it is possible to build a computable archimedean field such that *no* transcendence basis is c.e. In fact, we will show something stronger: that there is a computable archimedean field such that no transcendence basis even contains an infinite c.e. set.

Theorem 6.3. *There is a computable archimedean field such that every transcendence basis is immune. That is, no transcendence basis contains an infinite c.e. set.*

Proof. As in the proof of Theorem 6.1, we start with a computable archimedean field F which is a purely transcendental extension of $\mathbb{Q}$ with a computable pure transcendence basis. We then build a copy $\widehat{F}$ such that if W_e is infinite, then the first $e+1$ elements of W_e are algebraically dependent in $\widehat{F}$. Thus W_e can not be contained in any transcendence basis of $\widehat{F}$.

We use the same setup and notation as in the proof of Theorem 6.1. Requirements P_i, R_i and Q_s are identical. We replace the requirement D_n with

D_e: If W_e is infinite, then W_e is algebraically dependent.

Clearly satisfying all requirements results in a field with no c.e. transcendence basis.

<u>Construction:</u> We proceed as in the construction in the proof of Theorem 6.1, except for the first two steps on the construction at stage s. Instead, we do the following:

1. Check whether there is some $e < s$ for which D_e has not been satisfied, and for which $|W_{e,s}| \geq e+1$. If there is no such e, then set $f_{s+1} = f_s$ and skip to step 3. Otherwise, suppose the first $e+1$ elements of such a W_e are $b_{i_0}, \ldots, b_{i_e}$, with labels $p_{i_0}, \ldots, p_{i_e}$.

2. Let $\{x_{j_0}, \ldots, x_{j_n}\}$ be the set of variables appearing in $p_{i_0}, \ldots, p_{i_e}$. If $n < e$, then D_e is satisfied so set $f_{s+1} = f_s$ and skip to step 3. Otherwise, we have $e < n$. Set $m = e$ and $f_{s_m} = f_s$. As long as $m \leq n$ do the following:

(a) - (c) Same as in the construction in the proof of Theorem 6.1.

(d) If $m < n$, increment m and go back to step 2a. If $m = n$, then set $f_{s+1} = f_{s_m}$ and continue to step 3.

3 - 5. Identical to those of Theorem 6.1

This completes the construction.

Verification: We check that each requirement is satisfied by the construction. For each e, if W_e is infinite, then there will be a stage s for which $D_{e'}$ is satisfied for all $e' < e$, and for which $W_{e,s}$ contains $e + 1$ many elements. If needed, the construction redefines f_s so that the $e+1$ elements of $W_{e,s}$ all have labels with e or fewer distinct variables. Thus the $e + 1$ elements of $W_{e,s}$ are all in the algebraic span of e or fewer elements of $\widehat{F}$. Therefore, the elements of W_e cannot be algebraically independent, so D_e is satisfied. To see that P_i is satisfied for each i, note that the only $b_k \in \widehat{F}_s$ for which $f_s(b_k) \neq f_{s+1}(b_k)$ are ones whose labels include variables x_i with $i \geq e$ when we attempt to satisfy D_e. Similarly for requirements R_i, we have $f_s^{-1}(a_i) \neq f_{s+1}^{-1}(a_i)$ only when we try to satisfy D_e and the label for a_i contains x_j for $j < e$. So once D_e is satisfied, $f_s(b_k) = f_{s+1}(b_k)$ and $f_s^{-1}(a_i) = f_{s+1}^{-1}(a_i)$ for every b_k and a_i with labels containing only x_j and for $j < e$. Thus P_i and R_i are met for each i. Requirements Q_s are satisfied as before. This completes the verification, and the proof.

$\square$

7. Questions

Many questions remain in this subject, most noticeably whether there is a (nice) algebraic criterion on computable ordered fields which determines the computable dimension. Ordered fields with finite transcendence degree must be computably categorical, so here ordered fields are easier to analyze than fields in general. It appears though that the infinite transcendence degree case is no easier for ordered fields than non-ordered fields. However, it is unknown whether the algebraic criterion (whatever it might be) would be identical in the ordered and non-ordered cases.

The example given in [11] of a field with finite transcendence degree which is not computably categorical is a non-real field. Therefore we ask whether there is a formally real field (without an order specified) with finite transcendence degree which is not computably categorical.

Although archimedean fields must have computable dimension either 1 or ω, the same is not known for ordered fields in general. It is possible to build a non-archimedean ordered field with infinite transcendence degree which is computably categorical (as in Theorem 4.5). However, attempting a construction as we used in Theorem 4.2 to show a non-archimedian field has computable dimension ω yields problems beyond the possibility of transcendental elements being algebraically identifiable. Even if the isomorphism built between F and $\widehat{F}$ is not computable, there may be another isomorphism which is – unlike in the archimedian case, there will be many isomorphisms between the fields.

In Theorem 4.2 we required that the field possess a *computable* pure transcendence basis. It would be nice to eliminate this non-algebraic restriction. It is unclear whether or not this can be done. The proof still goes through if the pure transcendence basis is Π^0_1, however it is not clear that every ordered field possesses a *pure* transcendence basis which is Π^0_1. Thus we ask, are there computable purely transcendental ordered fields in which every pure transcendence basis has complexity greater than Π^0_1? If so, how can the proof of Theorem 4.2 be modified to apply to such fields?

Acknowledgements

Much of this article comes from the author's Ph.D. thesis at the University of Connecticut, under the supervision of D. Reed Solomon. Many thanks to Reed Solomon for his guidance and support, as well as Asher Kach, Joseph Miller and Russell Miller for insightful conversation.

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