

# Counting Liars and Truth-tellers: Binomial Identities through Logic Puzzles

Oscar Levin

University of Northern Colorado

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# Example Puzzle

Consider logic puzzles like the following.

## Three Trolls

You come upon three trolls guarding a bridge. You know each troll either always lies or always tells the truth. The trolls speak:

**Troll 1:** One of us is a liar.

**Troll 2:** No, one of us is a truth-teller.

**Troll 3:** We are all liars.

Which trolls are liars and which are truth-tellers?

The only solution is for Troll 2 to be telling the truth and the other trolls to be lying.

How many puzzles are there like this?

# Assumptions

## Definition

An  $n$ -troll puzzle satisfies the following:

- Each of the  $n$  trolls always speaks the truth or always lies.
- Each troll makes one statement of the form: exactly  $x$  of us are truth-tellers.
- The values for  $x$  range from 0 to  $n$ .
- We do not distinguish between the different orders in which the trolls could speak.

## Shorthand

We represent a puzzle with a  $n$ -tuple of numbers from 0 to  $n$ , written in increasing order. e.g. (0,2,2,4,5)

# Counting Puzzles

Choose  $n$  numbers from a collection of  $n + 1$ :  $\binom{n+1}{n}$

WRONG! here, repeats are allowed. E.g. (0, 2, 2, 4, 5)

Instead of  $n + 1$  choose  $n$ , we have  $n + 1$  multi-choose  $n$ :

$$\left( \binom{n+1}{n} \right)$$

# Multi-choose

One way to count: Where can we switch to the next higher number?

$$(0, 2, 2, 4, 5) \quad \iff \quad * || * * || * | *$$

$$(1, 3, 3, 3, 3) \quad \iff \quad | * || * * * * ||$$

We need to arrange  $n$  stars (the numbers) and  $n + 1 - 1$  bars (the switches). So

$$\left( \binom{n+1}{n} \right) = \binom{2n}{n} = \frac{(2n)!}{n!n!}$$

When  $n = 5$ , there are  $\left( \binom{6}{5} \right) = \binom{10}{5} = 252$  puzzles.

## Another multi-choose

Another way to count: divide into cases by the number of distinct statements.

Choose which statements and then where to switch:

1 statement:  $\binom{n+1}{1} \binom{n-1}{0}$  puzzles.

2 statements:  $\binom{n+1}{2} \binom{n-1}{1}$  puzzles

3 statements:  $\binom{n+1}{3} \binom{n-1}{2}$  puzzles.

$n$  statements:  $\binom{n+1}{n} \binom{n-1}{n-1}$  puzzles.

$$\text{Total: } \sum_{k=1}^n \binom{n+1}{k} \binom{n-1}{k-1}$$

# Combinatorial Corollary

$$\binom{2n}{n} = \sum_{k=1}^n \binom{n+1}{k} \binom{n-1}{k-1}$$

				1										
				1	1									
				1	2	1								
				1	3	3	1							
				1	4	6	4	1						
				1	5	10	10	5	1					
				1	6	15	20	15	6	1				
				1	7	21	35	35	21	7	1			
				1	8	28	56	70	56	28	8	1		
				1	9	36	84	126	126	84	36	9	1	
				1	10	45	120	210	252	210	120	45	10	1

# Solutions

## Definition

A *solution* is an assignment of truth values to the statements which is consistent with all the statements.

Some puzzles have no solution, some have multiple solutions.

## Theorem

*An assignment of truth values to the  $n$  statements is a solution iff there are exactly  $x$  statements of the form “ $x$  of us are truth-tellers” and those statements are assigned T and all others are assigned F.*

## Example

(1, 2, 2, 3, 4) has 3 solutions: FFFFF, TFFFF and FTTF

(0, 1, 1, 2, 3) has no solutions.

(0, 2, 2, 4, 5) has a unique solution: FTTF

# Counting solutions

How many  $n$ -troll puzzle-solution pairs are there?

Divide into cases by the number of truth-tellers:

0 truth-tellers:  $\binom{n}{n}$  puzzle-solution pairs

1 truth-teller:  $\binom{n}{n-1}$  puzzle-solution pairs

⋮

$n$  truth-tellers:  $\binom{n}{0}$  puzzle-solution pairs.

Total:  $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

## In other words...

Using a “stars and bars” argument:  $\binom{\binom{n}{k}}{k} = \binom{n+k-1}{k}$ .

$$\binom{\binom{n}{0}}{0} + \binom{\binom{n}{1}}{1} + \binom{\binom{n}{2}}{2} + \cdots + \binom{\binom{n}{n}}{n}$$

$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n}$$

By the Hockey-stick theorem:

$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n} = \binom{2n}{n}$$

$$\binom{\binom{n}{0}}{0} + \binom{\binom{n}{1}}{1} + \binom{\binom{n}{2}}{2} + \cdots + \binom{\binom{n}{n}}{n} = \binom{\binom{n+1}{n}}{n}$$

# Coincidence?

That looks familiar. As it turns out, this is also the number of  $n$ -troll puzzles.

## Theorem

*For any  $n$ , there are exactly as many  $n$ -troll puzzles as there are puzzle-solutions pairs.*

Strange... some puzzles have no solutions and some have multiple solutions, but everything evens out.

## If order matters:

Number of puzzles:  $(n + 1)^n$

Number of solutions:  $\binom{n}{0} + n\binom{n}{1} + n^2\binom{n}{2} + \dots + n^n\binom{n}{n}$   
 $= (n + 1)^n$

## If order doesn't matter, but only $k$ distinct statements:

Number of puzzles:  $\binom{n + 1}{k} \binom{n - 1}{k - 1}$

Number of solutions:  $\binom{n + 1}{k} \binom{n - 1}{k - 1}$

# Open Questions

1 Why?

2 How many  $n$ -troll puzzles have a unique solution?

# Thanks for listening

Slides available at  
[www.oscarlevin.com](http://www.oscarlevin.com)