Graphs between computable and highly computable

Oscar Levin

University of Northern Colorado

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Joint work with Matthew Jura and Tyler Markkanen

Computability and Graphs

A graph is computable provided there is an algorithm giving the edge relation.

$V = \mathbb{N}$. Is there an edge between vertices 7 and 253?

Compare: what vertices are adjacent to vertex 7?
This might be a harder question.

If a computable graph has a computable neighborhood relation, we say the graph is highly computable.

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Computable Chromatic Number

The computable chromatic number $\chi^c(G)$ is the smallest n for which there is a computable proper vertex coloring using n colors.

Theorem (Bean)

There is a computable planar graph G with $\chi(G) = 3$ but $\chi^c(G) = \infty$.

Theorem (Schmerl)

Every highly computable graph has $\chi^c(G) \leq 2\chi(G) - 1$ (and this bound is tight).

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Computable Sets

A subset of $\mathbb{N}$ is computable if there is some algorithm which determines membership.

Algorithm means... C++ program... or Turing machine. The algorithm is finite, but we have unlimited time and space.

$\{\varphi_e\}_{e \in \mathbb{N}}$ is a complete, effective list of all algorithms.

Non-Computable Sets

Not all sets are computable.

Example

$K = \{e \mid \varphi_e(e)\downarrow\}$ is not computable.

K is not computable, but it is computably enumerable (c.e.): there is an algorithm that lists all elements.

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Between Computable and K

K is Turing complete for the c.e. sets.

That is, if A is a c.e. set, then we can compute A using K.

$A \leq_T B$ means there is an oracle Turing machine Φ_e which computes A using oracle B. So $A = \Phi_e^B$.

Post's Problem: is there a non-computable c.e. set $A <_T K$?
Yes.

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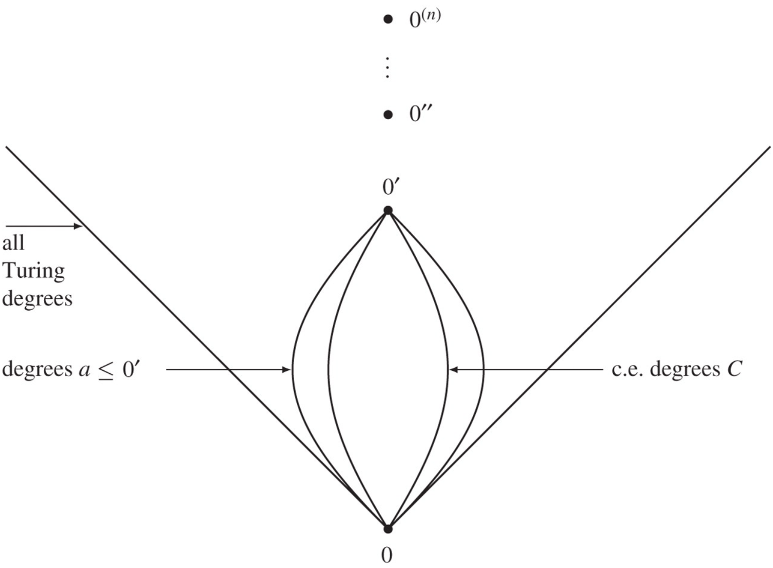
(A small part of) The Big Picture



A-computable Graphs

How hard is it to compute the neighborhood relation, N_G?

No matter what $N_G \leq_T K$. If G is highly computable, $N_G \leq_T \emptyset$.

But there are sets between $\emptyset$ and K.

Definition (Gasarch and Lee)

A computable graph G is A-computable provided $N_G \leq_T A$.

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Theorem (Gasarch and Lee)

For any non-computable c.e. set A, there exists an A-computable graph G with $\chi(G) = 3$ but $\chi^c(G) = \infty$

In other words: having more (but not complete) information about the neighborhood relation doesn't help.

The theorem generalizes to many (every?) graph property with different results for computable and highly computable graphs.

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What if A is not c.e.?

There are non-c.e. sets A between $\emptyset$ and K. What about them?

Theorem (JLM)

There exists a non-computable set $A \leq_T K$ such that every A-computable graph with chromatic number 3 has finite computable chromatic number.

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Proof Idea

Build A to be limit computable using a finite injury priority construction.

Essentially: for each potential A-computable graph, attempt to color (using the highly computable coloring algorithm). If something goes wrong, either use (a few) extra colors or change A to make the graph not A-computable.

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Using graphs to classify sets

A-computable graphs act like computable graphs	A-computable graphs act like highly computable graphs
$A = K$	$A = \emptyset$
All non-computable c.e. sets	Some non-computable non-c.e. sets
Some non-computable non-c.e. sets	A is low for graph neighborhood: every A-computable graph is highly computable.

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Some non-computable non-c.e. sets	A is <u>low for graph neighborhood</u>: every A-computable graph is highly computable.

Classification of sets low for graph neighborhood

Theorem (JLM)

Let $A \leq_T K$ be a non-computable set. The following are equivalent.

1. *A is low for graph neighborhood.*
2. *Every c.e. set $B \leq_T A$ is computable.*
3. *Every A-computable graph with finite chromatic number has finite computable chromatic number.*
4. *Every A-computable graph with an Euler path has a computable Euler path.*

Thanks

Slides and paper:

`math.oscarlevin.com/research.php`