(995 Open Questions about)
Graph Labelings and Computability Theory

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Joint Mathematics Meeting

Atlanta, GA

January 7, 2017

Joint work with Taylor McMillan

Graphs and Computability Theory

Classical graph theory:

Which graphs have which features?

Computable graph theory:

Which computable graphs have which computable features

Often:

Is there a computable graph with this feature, but no computable such feature.

Some examples

- There is a computable graph with chromatic number 3 but no computable 3-coloring (Bean).
- There is a computable graph with an Euler path but no computable Euler path (Bean)
- There is a computable graph with domatic number 3 but no computable domatic 3-partition (Jura-L-Markkanen)
- There is a computable planar graph with no computable planar embedding (L-McMillan).

Functions on vertices

Given a graph $G = (V, E)$ define a function $f : V \to \mathbb{N}$ given some constraints.

Example

If $uv \in E$, then $f(u) \neq f(v)$.
Proper vertex coloring.

Example

If $uv \in E$ then $\gcd(f(u), f(v)) = 1$.
Prime labeling.

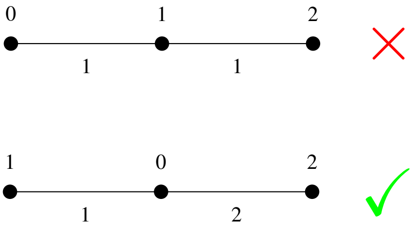
Example

If $uv \in E$ and $wx \in E$ then $|f(u) - f(v)| \neq |f(w) - f(x)|$.
Graceful labeling.

Labelings for Finite Graphs

Given a graph $G = (V, E)$:

A graceful labeling is an injection $f : V \to \{0, 1, \ldots, |E|\}$ such that the edge labels given by $|f(v_i) - f(v_j)|$ for $\{v_i, v_j\} \in E$ are distinct over all edges in G.



Labelings for Infinite Graphs

Consider (countably) infinite graphs ($V = \mathbb{N}$).

Graceful: $f : V \to \mathbb{N}$ must be a bijection. The induced edge labels must define a bijection between E and $\mathbb{Z}^+$ (usually).

Graceful Trees

Conjecture (Kotzig, Ringel and Rosa)

Every (finite) tree has a graceful labeling.

Theorem (Slater (1985); Chan, Cheung, Ng (2009))

Every infinite tree has a graceful labeling unless it has more than one but finitely many infinite degree vertices and does not contain an infinite path or once-subdivided infinite star.

Proof: use the infinite path (or other suitable feature) to ensure surjectivity, use large labels for other vertices.

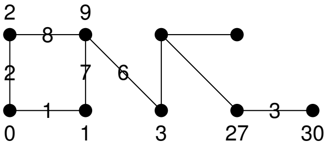
A classical result?

Theorem

Every infinite, locally finte graph has a graceful labeling.

Need:

1. Each natural number assigned to a vertex.
2. Each positive integer assigned to an edge.
3. No repeats.



What we really have…

Proposition

Every computable locally finite graph has a computable graceful labeling.

Conjecture

Every computable graph with a graceful labeling has a computable graceful labeling.

Other Questions

Basic question: Does every computable graph with a ____ labeling have a computable ____ labeling?

If no: replace “computable” with “highly computable.”

If yes: replace “computable” with “computably enumerable.”

Or: is the labeling uniformly computable?

Lots of questions

A few more types of labelings:

Edge-graceful
Odd-graceful
k-graceful
Skolem-graceful
Cordial
k-equitable
Hamming-graceful
Harmonious
Sequential
c-harmonious
Strongly c-harmonious
(k,d)-arithmetic
(k,d)-indexable
Elegant

Felicitous
Odd-harmonious
Even-harmonious
Vicious
Magic
Super edge-magic
Vertex-magic
H-magic
Distance magic
Antimagic
Face antimagic
Product antimagic
Sum graph
Prime
Vertex-prime

Radio
Line-graceful
k-sequential
Product cordial
Edge product cordial
Difference cordial
Prime cordial
Geometric
Mean
Irregular total
Set graceful
Set sequential
Square sum
Square difference

Change of perspective

Definition

A graph is a set E of 2-element subsets of $\mathbb{N}$
A computable graph is a computable set E of 2-elements subsets of $\mathbb{N}$.
A c.e. graph is a c.e. set E of 2-element subsets of $\mathbb{N}$.

A computable presentations of a particular graph is a computable graph isomorphic (as a graph) to that particular graph.

Definition

A computable (or c.e.) graph is graceful if $f : E \to \mathbb{Z}^+$ given by $f(\{x, y\}) = |x - y|$ is a bijection.

Translation

Proposition

Every locally finite computable graph is computably isomorphic to a computable graceful graph.

Conjecture

Every computable graph isomorphic to a graceful graph is computably isomorphic to a computable graceful graph.

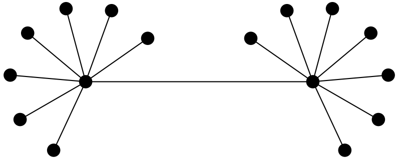
However...

. . . non-uniformly

Proposition

There is a computable sequence $E_0, E_1, \ldots$ of computable graphs, each isomorphic to a graceful graph, but no computable function φ_e is an isomorphism sending all graphs in the sequence to any graceful copy.

Idea: build each E_i to be a graph consisting of an infinite star and a finite star, connected by an edge:



Decision problem

How hard is it to decide whether a particular computable graph is graceful?

Proposition

Deciding whether a computable graph is graceful is Π^0_2-complete

Idea: Build a sequence of graphs $E_0, E_1, \ldots$ such that E_i is graceful iff $i \notin \mathrm{INF}$.

Each graph will be a pair of infinite stars with either a finite or an infinite path.

C.e. graphs

There are c.e. graphs that are not computable.

In fact, there are locally finite c.e. graphs with no computable presentations.

Proposition

Every graceful c.e. graph is computable.

Corollary

There is a c.e. graph isomorphic to a graceful graph but not isomorphic to any graceful c.e. graph.

Computable Categoricity

Definition

A computable graph is computably categorical provided any two computable isomorphic copies are *computably* isomorphic.

In general, it is difficult to compute an isomorphism between two isomorphic computable graphs.

Perhaps it is easier to build the isomorphism between two graceful computable graphs?

Nope (probably)

Conjecture

For any graceful graph, if any two graceful computable presentations of a graph are computably isomorphic, then any two computable presentations are computably isomorphic.

Open Question 996

Question

Is there a "natural" labeling which does help compute isomorphisms?

Thanks

Slides:

math.oscarlevin.com/research.php