Coloring Hypergraphs is Harder than Coloring Graphs

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Joint work with Conner Hatton

Computable Graphs and Coloring

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Coloring *connected* graphs:

- There is a computable 3-colorable graph with no computable 3-coloring.
- Every computable 2-colorable graph has a computable 2-coloring.

Hypergraphs and coloring

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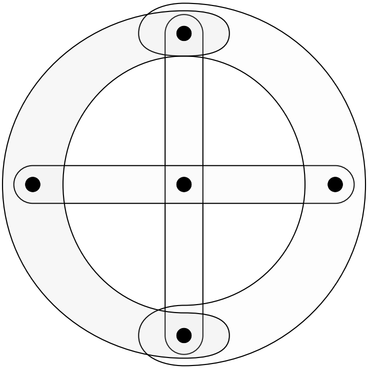
A *(weak) k-coloring* of a hypergraph is an assignment of k colors to vertices such that no hyperedge is monochromatic.

For example . . .



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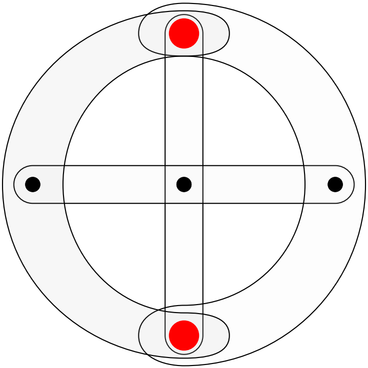

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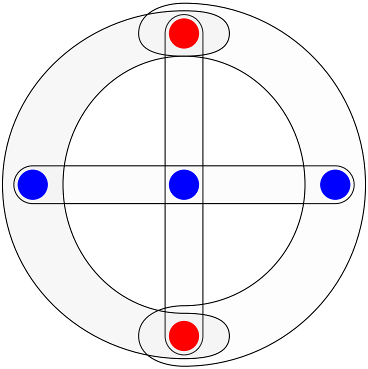

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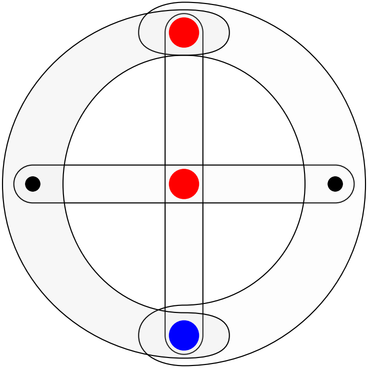

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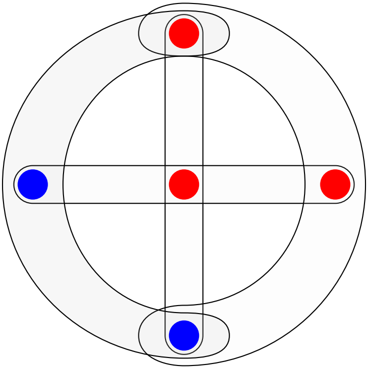

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Theorem

There exists a connected, computable 2-colorable hypergraph with no computable 2-coloring.

Construction

We build a computable hypergraph for which every potential 2-coloring is wrong.

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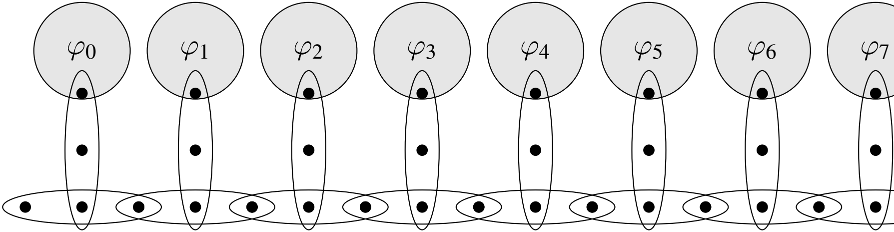
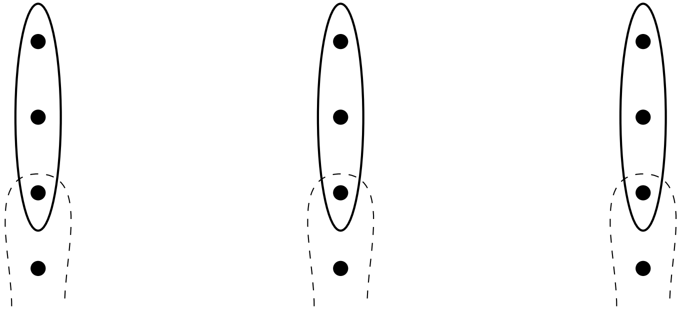


The gadgets



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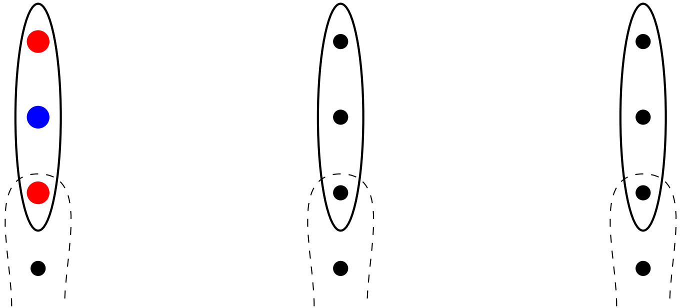

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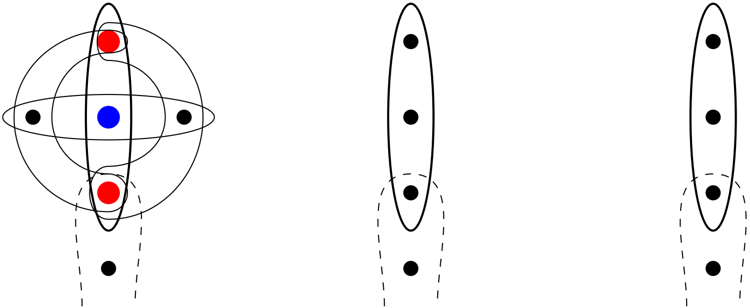

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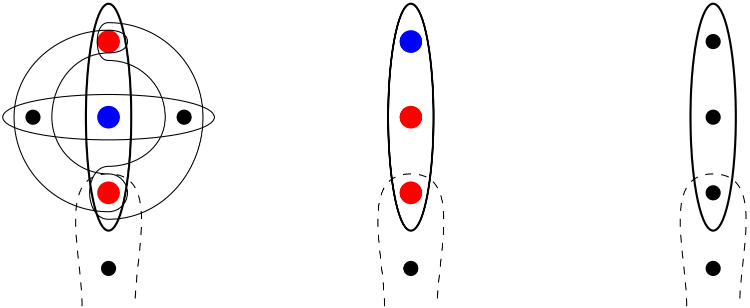

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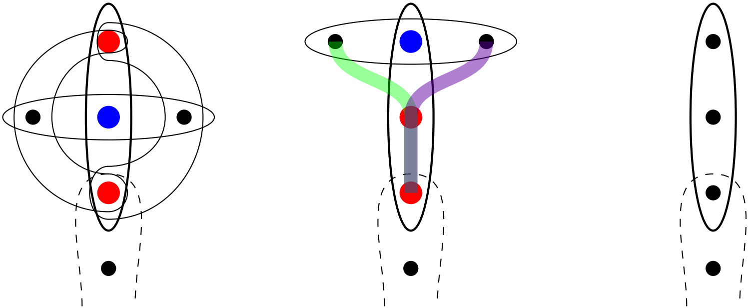

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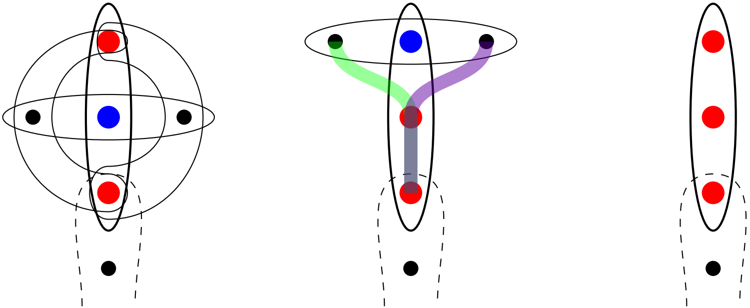

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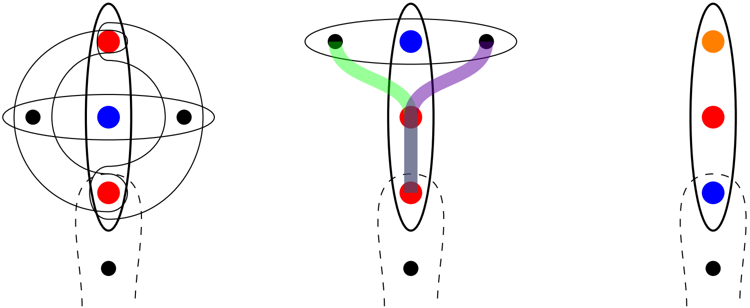

Increasing the computable chromatic number

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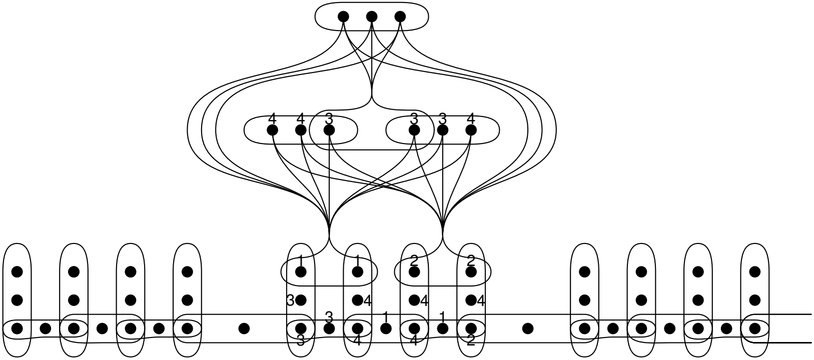

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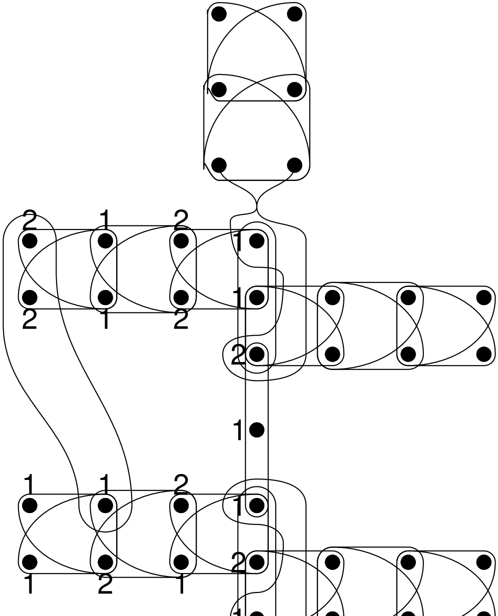
There is a connected, highly computable 2-colorable hypergraph with no computable 2-coloring.

The idea of a construction



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Consider not-necessarily connected graphs:

For 2-colorable graphs, you can compute a 2-coloring from the relation that says whether a pair of vertices is in the same component.

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For 2-colorable hypergraphs, even with this relation, you cannot compute a 2-coloring.

What should we color v_{207}? We don't know whether we have freedom to decide for ourselves!

Thanks!

Slides by email:

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