Knights and Knaves and Naive Set Theory

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*Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.*

Let $X = \{5, 6, 7, 8, 9\}$. Find a set $A \subseteq X$ with $|A| \in A$.

Paradox!

$$A = \{2, |A|\}$$

$$B = \{1, 3, |B|\}$$

$$C = \{1, 2, 3, 4, 5, 7, |C|\}$$

Puzzle!

Notation: $|A| = a$.

What is the cardinality of $A = \{2, 3, a\}$ (if it exists)?

Unique solution

What is the cardinality of $A = \{4, a, 2a\}$?

Two solutions

What is the cardinality of $A = \{1, 2, a, a-1\}$?

Three solutions

Unique Solution Cardinality Puzzles

Definition

A cardinality puzzle is a description of a set A that explicitly mentions the cardinality of A.

When does a cardinality puzzle have a unique solution?

$$A = \{3, 4, a, a+1, 2a-1\}$$

Consider the cases: $a = 2, 3, 4, 5$. Only one works: $A = \{3, 4, 5\}$.

Inverse Puzzle Problem

Which sets are the unique solution to a cardinality puzzle?

If we insist that a is listed as an element, then only selfish sets.

In fact...

Proposition

A is the unique solution to a cardinality puzzle iff A is selfish.

$$A = \{1, 5, 6, 10, 13, 42\}$$
$$A = \{1, 5, a, 10, 13, 42, f(a)\}$$

Where $f(a)$ is the line through $(6, 42)$ and $(7, 13)$

Knights and Knaves

$A = \{2, a\}$	$\cong$	I'm a knave
$A = \{1, a\}$	$\cong$	I'm a knight
$A = \{3, a\}$	$\cong$	X
???	$\cong$	He is a knave We are both knights

Symbiotic Sets

Let $|A| = a$ and $|B| = b$. Find the cardinalities:

$A = \{3, b\}$ Al: Bob is a knave.
$B = \{1, a, b\}$ Bob: We are both knights.

- Suppose $a = 1$. Then $b = 3$. But then $B = \{1, 1, 3\}$
- Suppose Al is a knave. This means Bob is a knight. But Bob's statement is false.
- Thus $a = 2$, so $B = \{1, 2, b\}$ and $b \neq 3$. So $b = 2$.
- Thus Al is a knight, so Bob is a knave (and indeed his statement is false).

What about Carl?

Al: Only one of us is a knave.

Bob: No, only one of us is a knight.

Carl: We are all knaves.

$$A = \{1, 3, 5, 6, 7, b, c - 7\} \qquad 5 \leq a \leq 7$$

$$B = \{7, 11, a, c\} \qquad 2 \leq b \leq 4$$

$$C = \{4, 7, 11, 12, 13, 14, 15, 16, a, b, c\} \qquad 8 \leq c \leq 11$$

A set "asserts" all its elements are distinct (its size is maximal).

Open Questions

- Does every knight and knave puzzle have a matching cardinality puzzle?
- Is the correspondence better suited for multi-valued logics? There are lots of ways for a set to “lie.”

Thanks!

Slides:

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