Computable Dimension of Ordered Fields

Oscar Levin

Department of Mathematics
University of Connecticut

Joint Mathematics Meeting
January 8th, 2009

Introduction

Main Question

Given a computable ordered field, how many computable copies are there, up to computable isomorphism?

Think: refine isomorphism classes to distinguish between ordered fields which are isomorphic but not computably isomorphic.

Alternatively: how many substantially different ways are there to computably code up an ordered field.

Computability

Intuitive definition

- A set of natural numbers is computable if there is an algorithm which decides whether any natural number is in the set.
- A function $f : \mathbb{N} \to \mathbb{N}$ is computable if there is an algorithm which gives $f(n)$ for all $n \in \mathbb{N}$.

Computable ordered fields

An ordered field is computable if the field operations ($+$ and $\cdot$) are computable functions, and the ordering ($\leq$) is a computable relation.

We take $\mathbb{N}$ to be the elements of the field.

Examples

$\mathbb{Q}$, $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, $\mathbb{Q}(\sqrt{p_i})_{i\in\mathbb{N}}$, $\mathbb{Q}(t)$, $\mathbb{Q}(t_i)_{i\in\mathbb{N}}$, etc.

Computable Dimension

Definition

- The *computable dimension* of a computable structure $\mathcal{A}$ is the number of computable copies of $\mathcal{A}$ up to computable isomorphism.
- A computable structure $\mathcal{A}$ is *computably categorical* if it's computable dimension is 1.

Theorem

Let F be a computable ordered field with finite transcendence degree. Then F is computably categorical. In fact, every isomorphism between copies of F is a computable isomorphism.

Towards the Proof

We have:

$$\psi : F \to \hat{F}$$

and need to compute $\psi(a)$ for each $a \in F$.

Easy example

Let $F = \mathbb{Q}$: Search through F and $\hat{F}$ until we find the 1 elements. Continue.

Slightly harder

Let $F = \mathbb{Q}(t_0, t_1, \ldots, t_n)$: Non-uniformly determine $\psi(t_0), \ldots \psi(t_n)$. Continue.

The General Case

In general, F can be an algebraic extension of $E = \mathbb{Q}(t_0, \ldots, t_n)$.

Each $a \in F$ is either in E or is the root of a polynomial in $E[x]$.

In the first case ($a \in E$): find $\psi(a)$ as before.

In the second case: search through a list of all polynomials in $E[x]$ to find one such that $p(a) = 0$.

Find the cooresponding polynomial $\hat{p}$ in $\hat{E}[x]$. Find a root of $\hat{p}$ in $\hat{F}$.

Problem: is it the *correct* root?

Distinguishing roots

Given a polynomial $p(x)$ and roots a and b, decide whether $a = b$.

Calculus method

Compare the signs of $\{p'(a), p''(a), \dots, p^{(n)}(a)\}$ and $\{p'(b), p''(b), \dots, p^{(n)}(b)\}$.

Logic method

Pass to the real closures of F and $\hat{F}$, and decide whether a and b are both the kth least root of $p(x)$, using the decidability of RCF.

Either way, we can determine if we have found the correct root. If not, find another one.

Fields with Infinite Transcendence Degree

The infinite transcendence degree case looks to be much harder.

Theorem

If F is a computable ordered field with infinite transcendence degree, then F has infinite computable dimension when:

- *F is real closed.*
- *F is archimedean, purely transcendental, with a computable pure transcendence basis.*

Problem: For fields between these, might the extra roots help match up transcendence bases?

Thanks for listening.

Slides available at OscarLevin.com