Counting Liars and Truth-tellers: Binomial Identities through Logic Puzzles

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Example Puzzle

Consider logic puzzles like the following.

Three Trolls

You come upon three trolls guarding a bridge. You know each troll either always lies or always tells the truth. The trolls speak:

> **Troll 1:** One of us is a liar.
> **Troll 2:** No, one of us is a truth-teller.
> **Troll 3:** We are all liars.

Which trolls are liars and which are truth-tellers?

The only solution is for Troll 2 to be telling the truth and the other trolls to be lying.

How many puzzles are there like this?

Assumptions

Definition

An n-troll puzzle satisfies the following:

- Each of the n trolls always speaks the truth or always lies.
- Each troll makes one statement of the form: exactly x of us are truth-tellers.
- The values for x range from 0 to n.
- We do not distinguish between the different orders in which the trolls could speak.

Shorthand

We represent a puzzle with a n-tuple of numbers from 0 to n, written in increasing order. e.g. (0,2,2,4,5)

Counting Puzzles

Choose n numbers from a collection of $n+1$: $\binom{n+1}{n}$

WRONG! here, repeats are allowed. E.g. $(0,2,2,4,5)$

Instead of $n+1$ choose n, we have $n+1$ multi-choose n:

$$\left(\!\!\binom{n+1}{n}\!\!\right)$$

Multi-choose

One way to count: Where can we switch to the next higher number?

$$(0, 2, 2, 4, 5) \qquad \Longleftrightarrow \qquad *||**||*|*$$

$$(1, 3, 3, 3, 3) \qquad \Longleftrightarrow \qquad |*||****||$$

We need to arrange n stars (the numbers) and $n + 1 - 1$ bars (the switches). So

$$\left(\!\!\binom{n+1}{n}\!\!\right) = \binom{2n}{n} = \frac{(2n)!}{n!n!}$$

When $n = 5$, there are $\left(\!\!\binom{6}{5}\!\!\right) = \binom{10}{5} = 252$ puzzles.

Another multi-choose

Another way to count: divide into cases by the number of distinct statements.

Choose which statements and then where to switch:
1 statement: $\binom{n+1}{1}\binom{n-1}{0}$ puzzles.
2 statements: $\binom{n+1}{2}\binom{n-1}{1}$ puzzles
3 statements: $\binom{n+1}{3}\binom{n-1}{2}$ puzzles.
n statements: $\binom{n+1}{n}\binom{n-1}{n-1}$ puzzles.

Total: $\displaystyle\sum_{k=1}^{n} \binom{n+1}{k}\binom{n-1}{k-1}$

Combinatorial Corollary

$$\binom{2n}{n} = \sum_{k=1}^{n} \binom{n+1}{k} \binom{n-1}{k-1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Solutions

Definition

A *solution* is an assignment of truth values to the statements which is consistent with all the statements.

Some puzzles have no solution, some have multiple solutions.

Theorem

An assignment of truth values to the n *statements is a solution iff there are exactly* x *statements of the form "*x *of us are truth-tellers" and those statements are assigned T and all others are assigned F.*

Example

$(1, 2, 2, 3, 4)$ has 3 solutions: FFFFF, TFFFF and FTTFF
$(0, 1, 1, 2, 3)$ has no solutions.
$(0, 2, 2, 4, 5)$ has a unique solution: FTTFF

Counting solutions

How many n-troll puzzle-solution pairs are there?

Divide into cases by the number of truth-tellers:

0 truth-tellers: $\left(\!\binom{n}{n}\!\right)$ puzzle-solution pairs

1 truth-teller: $\left(\!\binom{n}{n-1}\!\right)$ puzzle-solution pairs

$\vdots$

n truth-tellers: $\left(\!\binom{n}{0}\!\right)$ puzzle-solution pairs.

Total: $\left(\!\binom{n}{0}\!\right) + \left(\!\binom{n}{1}\!\right) + \left(\!\binom{n}{2}\!\right) + \cdots + \left(\!\binom{n}{n}\!\right)$

In other words...

Using a "stars and bars" argument: $\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$.

$$\left(\!\!\binom{n}{0}\!\!\right) + \left(\!\!\binom{n}{1}\!\!\right) + \left(\!\!\binom{n}{2}\!\!\right) + \cdots + \left(\!\!\binom{n}{n}\!\!\right)$$

$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n}$$

By the Hockey-stick theorem:

$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{2n-1}{n} = \binom{2n}{n}$$

$$\left(\!\!\binom{n}{0}\!\!\right) + \left(\!\!\binom{n}{1}\!\!\right) + \left(\!\!\binom{n}{2}\!\!\right) + \cdots + \left(\!\!\binom{n}{n}\!\!\right) = \left(\!\!\binom{n+1}{n}\!\!\right)$$

Coincidence?

That looks familiar. As it turns out, this is also the number of n-troll puzzles.

Theorem

For any n, there are exactly as many n-troll puzzles as there are puzzle-solutions pairs.

Strange... some puzzles have no solutions and some have multiple solutions, but everything evens out.

I think not

If order matters:

Number of puzzles: $(n+1)^n$

Number of solutions: $\binom{n}{0} + n\binom{n}{1} + n^2\binom{n}{2} + \cdots + n^n\binom{n}{n}$
$= (n+1)^n$

If order doesn't matter, but only k distinct statements:

Number of puzzles: $\binom{n+1}{k}\binom{n-1}{k-1}$

Number of solutions: $\binom{n+1}{k}\binom{n-1}{k-1}$

Open Questions

1. Why?

2. How many n-troll puzzles have a unique solution?

Thanks for listening

Slides available at

`www.oscarlevin.com`