The Complexity of Transcendence Bases in Computable Ordered Fields

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Joint Mathematics Meeting
Baltimore, MD
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Yup, Works for Ordered Fields Too

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Countable fields of characteristic 0.

Domain is $\mathbb{N}$.

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Examples

$$\mathbb{Q}$$

$$\mathbb{Q}(\sqrt{2}, \sqrt{3})$$

$$\mathbb{Q}(\sqrt{p_i})_{i \in \mathbb{N}}$$

$$\mathbb{Q}(e^{\sqrt{2}})$$

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Transcendence bases

Recall, a set $S \subseteq F$ is algebraically dependent if for some $n \in \mathbb{N}$ there is a nonzero polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ and distinct $s_1, \ldots, s_n \in S$ such that $p(s_1, \ldots, s_n) = 0$.

S is algebraically independent if it is not algebraically dependent.

A maximal algebraically independent set in F is called a transcendence basis for F over $\mathbb{Q}$.

If F is an extension of $\mathbb{Q}$ and has a transcendence basis S, then F is algebraic over the field $\mathbb{Q}(S)$.

The field $\mathbb{Q}(S)$ is a purely transcendental extension of $\mathbb{Q}$, with a pure transcendence basis S.

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Some questions

Given S in a computable ordered field F,

1. how hard is it to decide whether S is algebraically dependent?
2. how hard is it to decide whether S is a transcendence basis?
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Given a computable ordered field F,

1. must there be a computable transcendence basis?
2. must there be a computable copy of F with a computable transcendence basis?
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How you could find a transcendence basis

First guess: every element of F is in the transcendence basis.

Wrong: this or that element is in $\mathbb{Q}$, so throw them out.

Still too much: some sets are algebraically dependent, so throw out an element from the set.

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Feilds with no c.e. trancendence basis

Build a copy of a purely transcendental field.

Diagonalize against all infinite c.e. sets.

When W_e enumerates enough elements, make one algebraically dependent on the others.

In fact, make one of the elements in W_e rational.

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Why that is difficult

Sounds easy, but remember we are building the field to be computable.

So after mentioning an element, we also need to say how it relates to other elements.

By the time we want to make an element of W_e rational, we might have already listed it as a transcendental and specified how it adds and multiplies with other elements.

For a field, we can always find a ”very large” rational, bigger than anything we have used, to assign this previously transcendental element.

For an ordered field, we don’t have this luxury.

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Nudging transcendentals

If we want to make a_k rational, but $a_7 < a_k < a_4$, then we need to pick a rational between a_7 and a_4.

But we also must consider where $a_k + a_4$ and $a_k \cdot a_4$ sit w.r.t. other elements. And $a_k^2 + a_7 \cdot a_4$. Etc.

This can be done. But the argument requires a purely transcendental ordered field so we can enumerate rationals and the transcendence basis, close under $+$ and $\cdot$ and get everything.

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Other ordered fields

Does every computable ordered field with infinite transcendence degree have a copy with no computable transcendence degree?

No.

Say x is transcendental. If we adjoin $y = \sqrt[3]{1 - x^3}$, we get a field in which $x^3 + y^3 = 1$ has non-trivial solutions.

But not very many. In fact, there might be only 1 such y between 7 and 8. So now we cannot nudge x over.

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