A Paradox of Finite Cardinality

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What is a set?

What do we tell our students?

Defining Sets

Axiom 0: Set Existence.

$$\exists x(x = x)$$

Axiom 1: Extensionality.

$$\forall z(z \in x \leftrightarrow z \in y) \quad \rightarrow \quad x = y$$

Axiom 2: Foundation.

$$\exists y(y \in x) \quad \rightarrow \quad \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))$$

Axiom 3: Comprehension Scheme.[1]

$$\exists y \forall x(x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x))$$

Axiom 4: Pairing.

$$\exists z(x \in z \wedge y \in z)$$

Axiom 5: Union.

$$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathcal{F} \quad \rightarrow \quad x \in A)$$

Axiom 6: Replacement Scheme.[2]

$$\forall x \in A \, \exists! y \, \varphi(x, y) \quad \rightarrow \quad \exists B \forall x \in A \, \exists y \in B \, \varphi(x, y)$$

Axiom 7: Infinity.

$$\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$$

Axiom 8: Power Set.

$$\exists y \forall z(z \subseteq x \quad \rightarrow \quad z \in y)$$

Axiom 9: Choice.

Defining Sets Naively

Definition

A set is an unordered collection of objects.

Examples:

$$\{0, 1, 2\}$$

$$\{0, 1, 2, \ldots, 10\}$$

$$\{0, 1, 2, \ldots\} = \mathbb{N}$$

$$\{x \in \mathbb{N} \ : \ x \text{ is even}\}$$

$$\{X \in \mathbb{N} \ : \ X \notin X\}$$

Careful!

$$\{0, 1, 2\} = \{2, 0, 1\}$$

$$\{0, 1, 2\} \neq \{0, 1, 2, 3\}$$

$$\{0, 1, 2\} \neq \{1, 2, 3\}$$

$$\{0, 1, 2\} = \{1 - 1, 1, 1 + 1\}$$

$$\{0, 1, 2\} = \{0, 1, 1 + 1, 2 - 1, 3 - 2\}$$

Sum more sets

Define $A + B = \{a + b \ : \ a \in A, b \in B\}$.

Is $|A + B| = \ \geq |A| + \ \cdot |B| - 1$?

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$?

$$A+B = \{0+1,\ 0+2,\ 0+3,\ 1+1,\ 1+2,\ 1+3,\ 2+1,\ 2+2,\ 2+3\}$$

$$A + B = \{1, 2, 3, 2, 3, 4, 3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

Sets for Counting

How many subsets of $A = \{0, 1, \ldots, 9\}$ are there?

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \ldots \{0, 1\}, \ldots, A\}$$

Let $A = \{0, 1, \ldots, 9\}$. Define $\mathcal{B}_2 = \{B \subseteq A \ : \ |B| = 2\}$. Find $|\mathcal{B}_2|$.

1996 Putnam Exam, B1

Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

- $n = 1$: $\{1\}$
- $n = 2$: $\{1\}$
- $n = 3$: $\{1\}, \{2, 3\}$
- $n = 4$: $\{1\}, \{2, 3\}, \{2, 4\}$
- $n = 5$: $\{1\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4, 5\}$

Sets containing their own cardinality

$$A = \{2, |A|\}$$

Other examples:

- $B = \{1, 3, |B|\}$
- $C = \{1, |D|\}$ where $D = \{2, |C|\}$

Comprehension

Axiom 3: Comprehension Scheme[1]

$$\exists y \forall x (x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x))$$

$$Y = \{x \in Z \ : \ \varphi(x)\} \text{ exists}$$

Let $\varphi(x)$ be "$x = 2 \vee x = |Y|$"

$$Y = \{x \in \mathbb{N} \ : \ x = 2 \vee x = |Y|\}$$

With the footnote

Axiom 3: Comprehension Scheme.
For each formula φ without y free:

$$\exists y \forall x (x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x))$$

But where is the fun in that!

Sometimes its OK

Examples:

$$A = \{|A|\}$$

$$B = \{2, 3, |B|\}$$

$$C = \{3, 4, 5, |C|,\ |C| + 1\}$$

$$D = \{x \in \mathbb{N}\ :\ x \leq |D|\}$$

Sometimes its more than OK

(i.e., NOT OK)

$$A = \{1, |A|\}$$

$$B = \{2, 4, |B|\}$$

$$C = \{3, 4, |C|,\ |C| + 1,\ |C| + 2\}$$

$$D = \{x \in \mathbb{N}\ :\ x < |D|\}$$

$|A|$ more:

Let $A = \{x \in \mathbb{N} \; : \; x \leq |B|\}$ where $B = \{x \in \mathbb{N} \; : \; |A| \leq x \leq 10\}$.

Find A and B.

Thanks!

Slides:

math.oscarlevin.com/research.php