Reverse Mathematics of Maximal and Perfect Matchings

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Joint work with Stephen Flood, Matthew Jura, and Tyler Markkanen.

Matchings in Graphs

Definition

A *matching* in a graph $G = (V, E)$ is a subset $M \subseteq E$ for which each vertex is incident to at most one edge in M.

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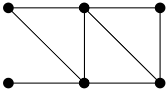

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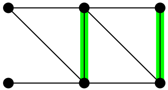

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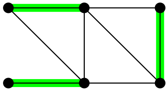

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Improving matchings

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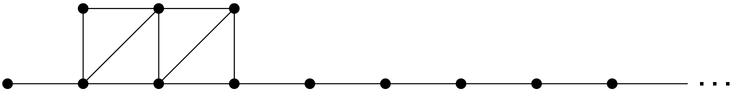

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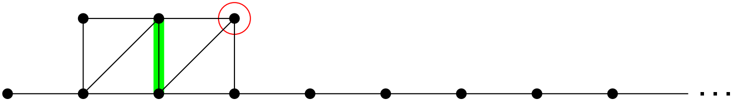

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An M*-alternating* path has edges alternating in and out of M.

An M*-augmenting* path is an M-alternating path that starts with an unmatched vertex and either ends in another unmatched vertex or is infinite.

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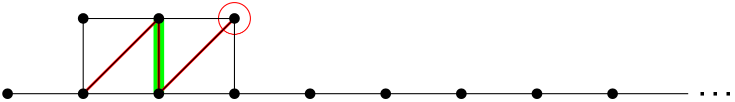

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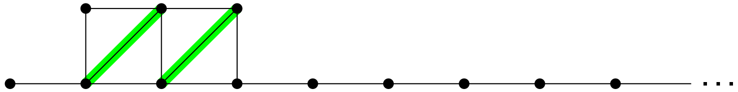

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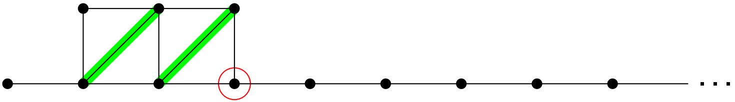

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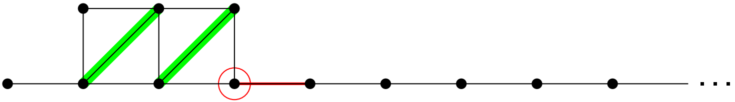

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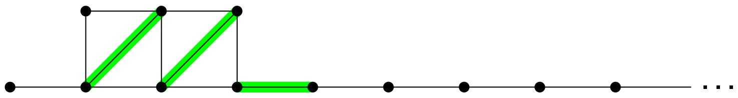

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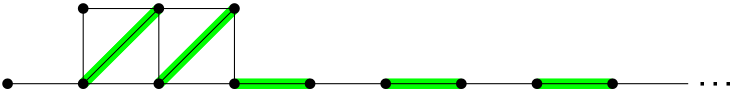

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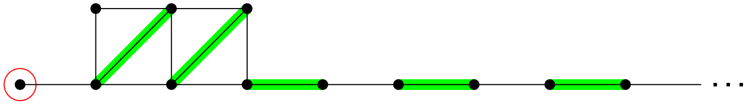

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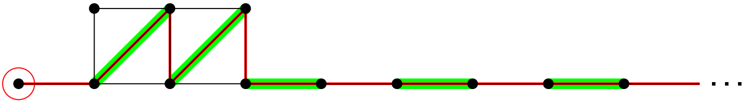

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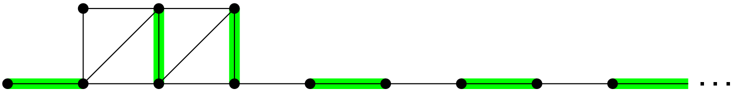

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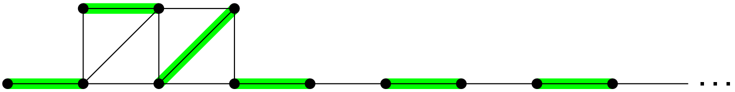

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Really, maximality seems to be a corollary to the proof of this theorem.

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Use Reverse Mathematics to understand the strength of both of these theorems.

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The plan:

1. Complete classification for locally finite graphs.
2. Get a sense why the general case is much much much much harder to classify (probably).

Locally finite and bounded graphs

Definition

- A graph is *locally finite* provided every vertex has finite degree.
- A graph is *bounded* provided there is a function $h : V \to \mathbb{N}$ s.t. $\forall x, y \in V(\{x, y\} \in E \to h(x) \geq y)$.

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Think: bounded = highly computable.

Locally finite and bounded graphs

Theorem

The following are equivalent over RCA_0*:*

1. *Every locally finite graph has a maximal matching.*
2. *A locally finite graph has a perfect matching iff it satisfies condition (A).*
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The following are equivalent over RCA_0*:*

1. *Every bounded graph has a maximal matching.*
2. *A bounded graph has a perfect matching iff it satisfies condition (A).*
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Proofs

Idea: Build a tree whose paths give perfect matchings.

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$\langle a_0, a_1, \ldots, a_n \rangle \in T$ iff $\{(0, a_0), (1, a_1), \ldots, (n, a_n)\}$ is a matching.

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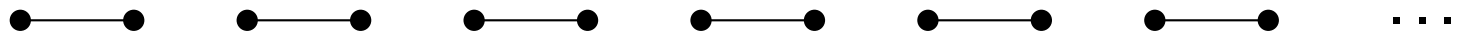
Condition (A) guarantees the tree will be infinite.

Reversals

PM for locally finite graphs implies ACA_0:



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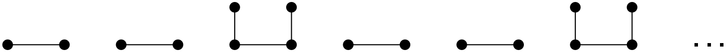

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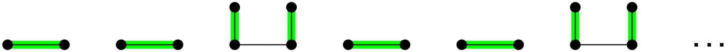

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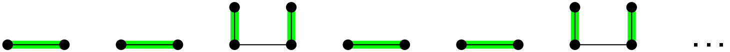

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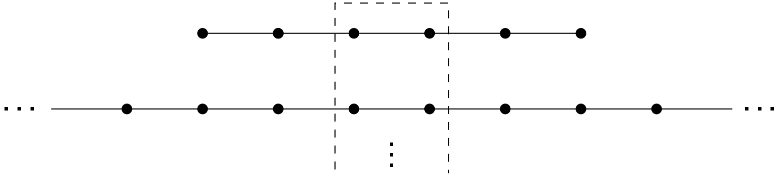

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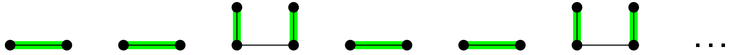

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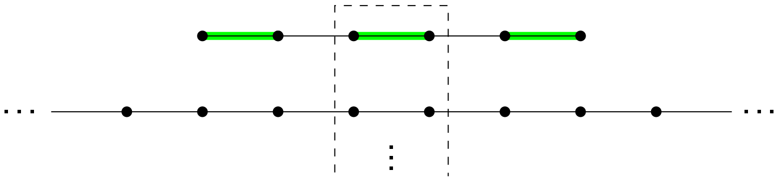

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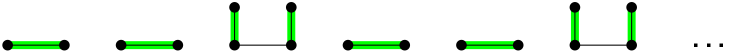

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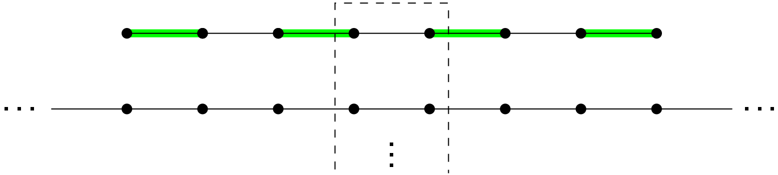

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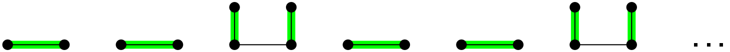

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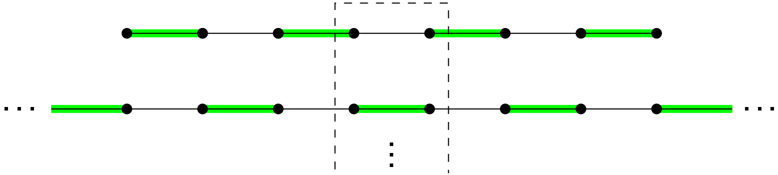

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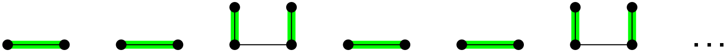

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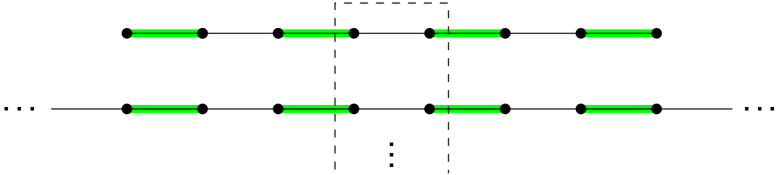

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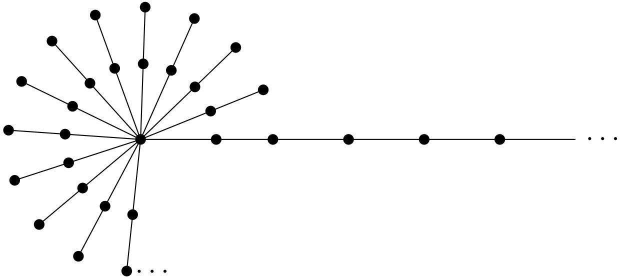

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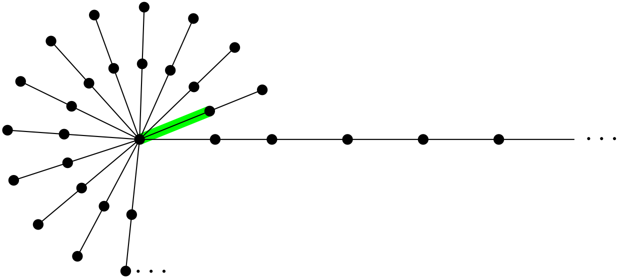

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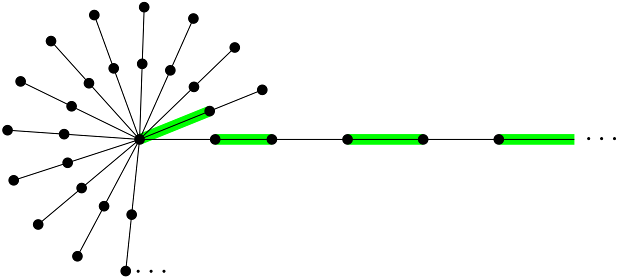

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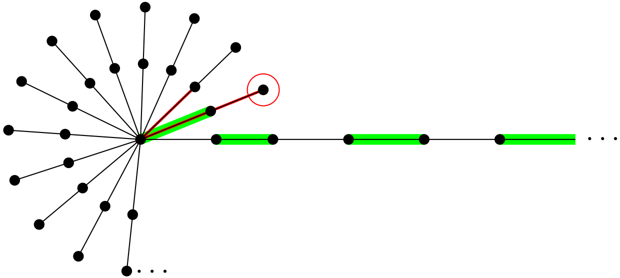

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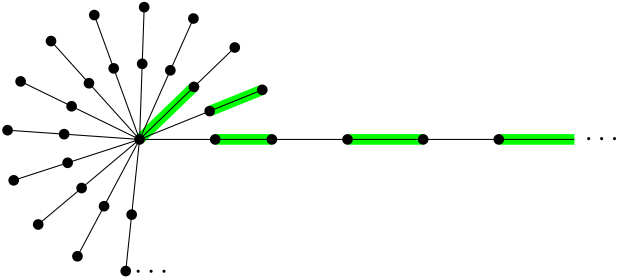

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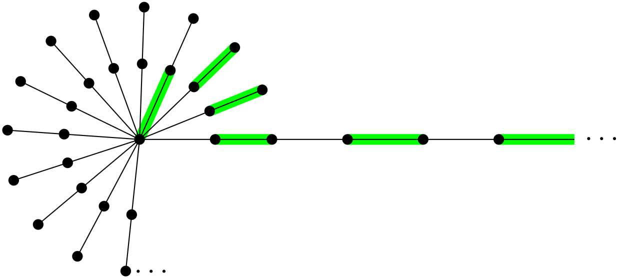

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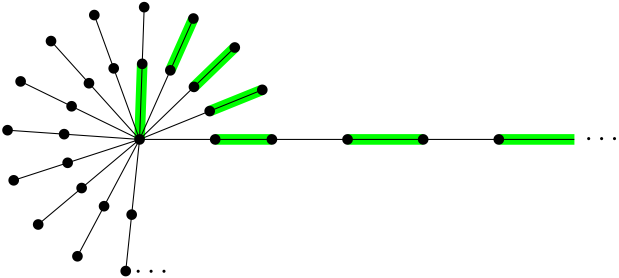

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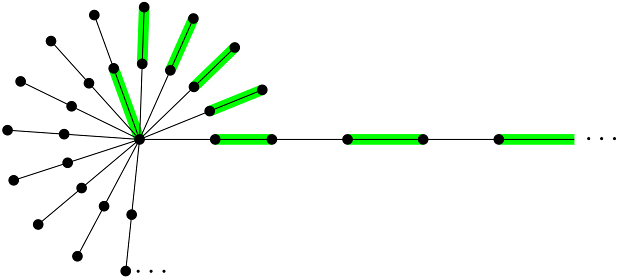

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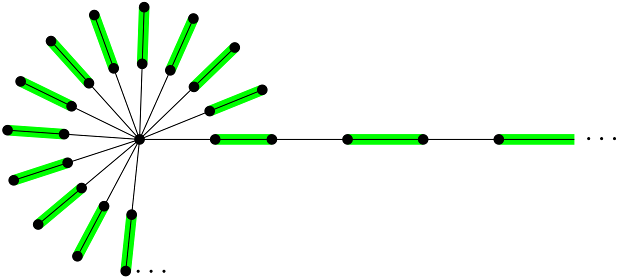

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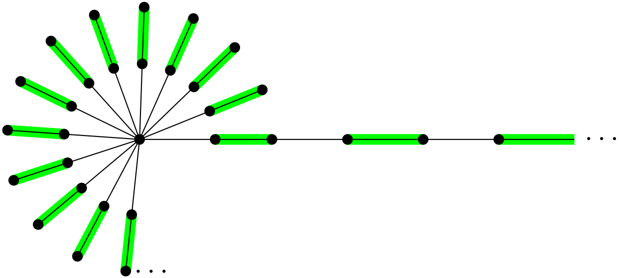

The problem: To use a larger matching, you must abandon a smaller matching.

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Zorn's lemma.

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Note: this is a proof in Π^1_2-CA.

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A proof of maximality

Take a maximal independent subgraph H of G.

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But the perfect matching would be independent in G, giving a larger independent matching. So any perfect matching of H is a maximal matching of G.

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If a graph contains an edge, then the graph contains a non-empty independent matching.

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What is the strength of this lemma? No idea!

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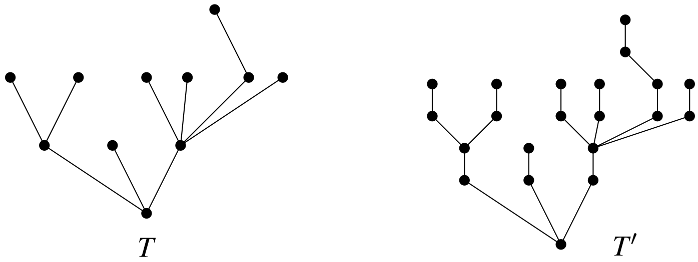

Proposition

MaxM implies Π^1_1-CA$_0$. PM implies Σ^1_1-AC$_0$.

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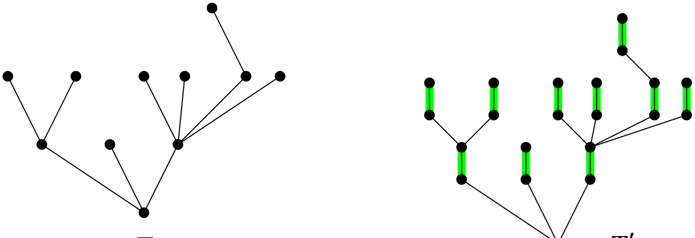

T $\quad$ T'

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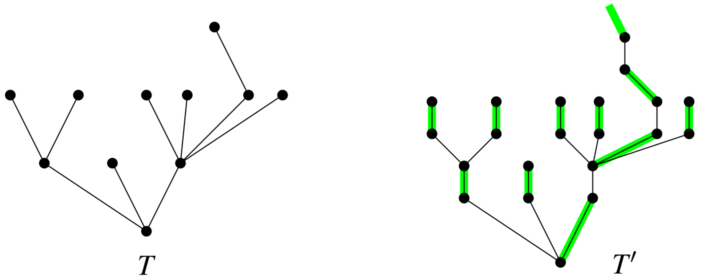

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Can we do better?

For any computable ordinal α, there is a computable graph G that satisfies condition (A), any perfect matching of which computes $\mathbf{0}^{(\alpha)}$.

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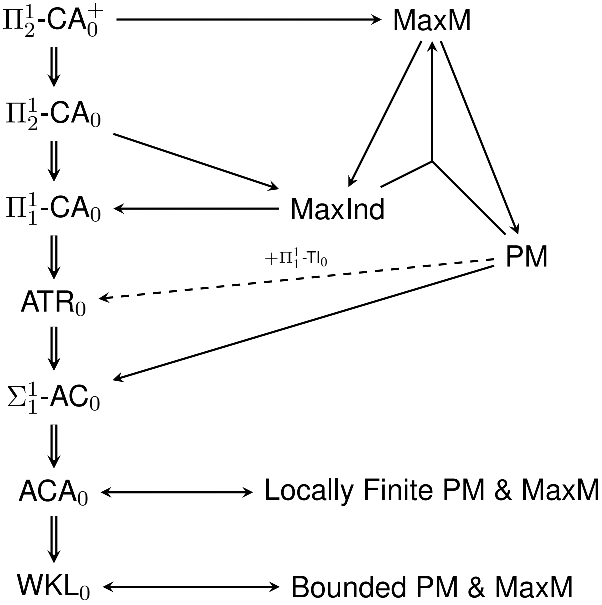
This would be enough to prove ATR_0 from PM, except we don't know how to prove G satisfies condition (A) without using Π^1_1-TI_0.
(Σ^1_1-DC_0)

The current picture



The End

Thanks!